Assigned: Wednesday, October 31, 2001
Due: Wednesday, November 7, 2001
Reading: Ross, Chapter 5 and Chapter 6
Noncredit Exercises: Ross, Chapter 5: Problems 15-38; Chapter 6: Problems 1, 8-15, 20-23
Problems:

1. [Read Example 3d on pp. 198-199 first.] Let the (straight) line segment ACB be a diameter of a circle of unit radius and center C. Consider an arc AD of the circle where the length **X** of the *arc* (measured clockwise around the circle) is a random variable uniformly distributed on $[0,2\pi)$. Now consider the "random chord" AD.

(a) Find the probability that the length **L** of the random chord is greater than the side of the equilateral triangle inscribed in the circle.

(b) Express **L** as a function of the random variable **X**, and find the probability density function for **L**.

2. The random variable **X** has probability density function $f_{\mathbf{X}}(u) = \begin{cases} 2(1-u), & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

Let $\mathbf{Y} = (1 - \mathbf{X})^2$.

(a) What is the CDF $F_{\mathbf{Y}}(v)$ of the random variable **Y**? Be sure to specify the value of $F_{\mathbf{Y}}(v)$ for all v, $-\infty < v < \infty$.

(b) Show that the $F_{\mathbf{Y}}(v)$ that you found in part (b) is a nondecreasing continuous function.

3. The radius of a sphere is a random variable **R** with pdf $f_{\mathbf{R}}(\rho) = \begin{cases} 3\rho^2, & 0 < \rho < 1, \\ 0 & \text{elsewhere.} \end{cases}$

(a) Use LOTUS to find the average radius, average volume and average surface area of the sphere. Does a sphere of average radius have average volume? Does a sphere of average radius have average surface area?

(b) Find the CDF $F_{\mathbf{V}}(\alpha)$ and pdf $f_{\mathbf{V}}(\alpha)$ of **V**, the volume of the sphere.

(c) Find E[**V**] directly from this pdf. Do you get the same answer as in part (a)? Why not?

(d) If the sphere is made of metal and carries an electrical charge of Q coulombs, what is the CDF $F_{\mathbf{S}}(x)$ and the pdf $f_{\mathbf{S}}(x)$ of the surface charge density **S** on the sphere ?

4. ["Give me an A! Give me a D! Give me a converter! What have you got? An A/D converter! Go Team!"]
A signal **X** is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value **Y** (where $\mathbf{Y} = \alpha$ if $\mathbf{X} > 0$ and $\mathbf{Y} = -\alpha$ if $\mathbf{X} \le 0$) is used. Note that **Y** is a *discrete* random variable.

(a) What is the pmf of **Y**?

(b) Suppose that $\alpha = 1$. If the signal **X** happens to have value 1.29, what is the error made in representing **X** by **Y**? What is the squared-error? Repeat for the case when **X** happens to have value $\pi/4$ and when **X** happens to have value $-\pi/4$.

(c) We wish to design the quantizer so as to minimize the squared-error. However, since **X** (and **Y**) are random, we can only minimize the squared-error in the probabilistic (that is, average) sense. Now, part (b) shows that the squared-error depends on the value of **X**, and can be expressed as $\mathbf{Z} = (\mathbf{X} - \mathbf{Y})^2 = g(\mathbf{X}) = \begin{cases} (\mathbf{X} - \alpha)^2 & \text{if } \mathbf{X} > 0 \\ (\mathbf{X} + \alpha)^2 & \text{if } \mathbf{X} \le 0. \end{cases}$

So we want to choose α so that E[**Z**] is as small as possible. Use LOTUS to e-zily find E[**Z**] as a function of α, and then find the value of α that minimizes E[**Z**].

(d) We now get more ambitious and use a 3-bit A/D converter which first quantizes **X** to the nearest integer **W** in the range –3 to +3. Thus, $\mathbf{W} = 3$ if $\mathbf{X} \ge 2.5$, $\mathbf{W} = 2$ if $1.5 \le \mathbf{X} < 2.5$, etc. Note that **W** is a discrete random variable. Find the pmf of **W**.

(e) The output of the A/D converter is a 3-bit 2's complement representation of **W**. Suppose that the output is ($\mathbf{Z}_2$, $\mathbf{Z}_1$, $\mathbf{Z}_0$). What is the pmf of $\mathbf{Z}_2$? of $\mathbf{Z}_1$? of $\mathbf{Z}_0$?

(f) **Noncredit exercise (but a real-life engineering problem!):** Suppose that **W** takes on values $-3\alpha, -2\alpha, -\alpha, 0, +\alpha, +2\alpha, +3\alpha$ and quantization is as before: **X** is mapped to the nearest **W** value. What value of α minimizes $E[(\mathbf{X} - \mathbf{W})^2]$?

5. The lifetime of a system with hazard rate $\lambda(t) = bt$ is a Rayleigh random variable **X** with pdf $f(u) = (bu)\bullet\exp(-bu^2/2)$ for $u > 0$ (Ross, p. 216). The system fails at time t, i.e. $\mathbf{X} = t$ is observed to have occurred on this trial. What is the maximum-likelihood estimate of the parameter b that occurs in the pdf and hazard rate? Remember that the maximum-likelihood estimate $\hat{b}$ of the parameter b maximizes the pdf at the observed value t. Thus, for given t, what value of b maximizes $(bt)\exp(-bt^2/2)$?

6. If hypothesis H_0 is true, the pdf of **X** is $f_0(u) = (1/2)\exp(-|u+1|)$, $-\infty < u < \infty$, while if hypothesis H_1 is true, the pdf of **X** is $f_1(u) = (1/2)\exp(-|u-1|)$, $-\infty < u < \infty$. Such pdfs are called LaPlacian or double exponential pdfs.

(a) Sketch the two pdfs.
Be careful: those absolute-value signs are trickier than they look!

(b) State the maximum-likelihood decision rule in terms of a threshold test on the observed value u of the random variable **X** instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_0(u)$ with 1.

(c) What are the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part (b)?

(d) Compute the values of the likelihood ratio for $u = -1.2, -1, -0.8, \ldots, 0.8, 1, 1.2$.

(e) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to (π_0/π_1). Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable **X**.

(f) If $\pi_0 = 2\pi_1$, what is the average probability of error of the Bayesian decision rule?

(g) What is the average error probability of a decision rule that always decides H_0 is the true hypothesis, regardless of the value taken on by **X**?

(h) Show that if $\pi_0 > e^2/(e^2+1)$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by **X**.

7. The random variable **X** models a physical parameter. If hypothesis H_0 is true, then, $f_0(u)$, the pdf of **X**, is Gaussian with mean 0 and variance a^2. On the other hand, if hypothesis H_1 is true, then $f_1(u)$, the pdf of **X**, is Gaussian with mean 0 and variance $b^2 > a^2$.

(a) Suppose that H_0 and H_1 have equal probability. Thus, for i = 0, 1, the pdf of **X** when hypothesis H_i is true can be thought of as the *conditional* pdf of **X** given that H_i occurred, i.e. $f_{\mathbf{X}|H_i}(u|H_i)$. Write an expression for the *unconditional* pdf of **X**. Is the unconditional pdf of **X** a Gaussian pdf?

(b) What is the likelihood ratio? Simplify your answer.

(c) What is the maximum-likelihood decision rule, and what are the false alarm probability and the missed detection probability of this rule?