Assigned: Wednesday, October 24, 2001
Due: Wednesday, October 31, 2001
Reading: Ross, Chapter 5, Chapter 9.1
Noncredit Exercises: Ross, Chapter 5: Problems 11, 13, 15-38; Theoretical Exercises 12-16, 21, 26, 28-30

Problems:

1. Consider a Poisson process with arrival rate λ.

(a) What is the mean number of arrivals in the interval (0, 4] ? That is, what is $E[\mathbf{N}(0,4]]$?

(b) What is $P[\{\mathbf{N}(0, 3] = 3\} \cap \{\mathbf{N}(2, 6] = 0\}]$?

(c) If we observe that there were 5 arrivals in (0, 6], what is the maximum-likelihood estimate of the arrival rate λ?

(d) Now suppose that $\lambda = \ln 2$. What is the probability that at least one arrival occurs in (0, t]?

2. A slotted ALOHA* communication system consists of a *very large number* of transmitters that *infrequently* send data packets to a single receiver. Transmissions are synchronized so that, regardless of the distances from the transmitters to the receiver, packets always begin to arrive at the receiver at the start of a time interval called a slot, and each packet lasts for the duration of the slot. If only one of the transmitters sends a packet during a slot, this transmission is successful, and the receiver sends an acknowledgment (ACK) signal. If two or more transmitters send packets in the same slot (say, slot #n), all the packets *collide* and none of the transmissions is successful. The receiver detects this collision and broadcasts a NACK (no acknowledgment) signal. Those who just transmitted thus know that their packets did not get through. These transmitters are said to be *backlogged.* All other transmitters also hear the NACK and are prohibited from transmitting until the *collision is resolved,* that is, until *all* the backlogged transmitters have managed to (successfully) re-transmit their packets. Each backlogged transmitter rolls a fair die and re-transmits its packet in slot #(n+i), where i denotes the outcome on the die that it rolled.

(a) What would happen if all the transmitters who became backlogged during slot #n didn't roll dice but simply immediately repeated their packet transmissions in slot #(n+1)?

(b) Suppose that two transmitters become backlogged during slot #n. Consider slot #(n+1).
What is the probability that no backlogged transmitter sends a packet in this slot?
What is the probability that a successful re-transmission occurs in this slot?
What is the probability that both transmitters re-transmit and collide once again in this slot?

(c) If one of the transmitters is successful in sending its packet during slot #(n+1), the other is guaranteed success in a future slot. In case of a collision in slot #(n+1), both transmitters toss their dice again and begin the process anew. Suppose that the collision is resolved in slot $\#(n + \mathbf{X})$, that is, the second successful re-transmission occurs in slot $\#(n + \mathbf{X})$. What are the values of $P\{\mathbf{X} = 2\}$, $P\{\mathbf{X} = 3\}$ and $P\{\mathbf{X} = 4\}$? **Warning:** computing the values of $P\{\mathbf{X} = 3\}$and $P\{\mathbf{X} = 4\}$ is not as straightforward as computing $P\{\mathbf{X} = 2\}$!

(d) Repeat parts (b) and (c) for the case of three transmitters becoming backlogged during slot #n. Assume that in the case of a second collision in slot #(n+1), all three transmitters re-toss their dice (even if only two transmitters collided).
For the rest of this problem, ignore backlogs and collision resolution entirely, and assume that packets that are not successful are lost forever and never re-transmitted. Thus, we are only concerned with *new* packets.

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* so called because it was invented at the University of Hawaii for connecting the computer terminals on their campus to a central mainframe. The actual slotted ALOHA collision resolution algorithm is too difficult to analyze in homework in this course. The method described above is simpler to analyze, but does not capture most of the nuances. A more detailed (and correct!) description of the actual system is often taught in the course ECE/CS 338 *Communication Networks for Computers* (which has ECE 313 as one of its prerequisites).

(e) The number of *new* packets that are transmitted in a slot is usually modeled as a Poisson random variable with parameter λ. Explain briefly what reasoning might have been used to justify this model, and what value should be ascribed to λ.

(f) Let p denote the probability that a successful new packet transmission occurs in a slot. Express p as a function of λ. The number of successful new packet transmissions in a slot is a Bernoulli random variable **Y** with parameter p. What value of λ maximizes E[**Y**]?

3. Do **either** part (a) **or** part (b). Then do parts (c)–(e).

(a) Attach to your homework a **photocopy** of your calculator's manual page(s) that explains which **formula** your calculator uses to compute Q(x). Reading the page might help too! Note: I **do not want** to know **which buttons** you have to press in order to find Q(x); I **want** to know **what formula** your calculator uses internally to find Q(x).
The xerographically–challenged are permitted to just copy the relevant formulas to their homework. **NEXT:** press the appropriate buttons to **find Q(5).**
If your calculator cannot compute Q(x), or if the manual does not state what formula is used to calculate Q(x) but just tells you which buttons to press, or if you have lost the manual, do part (b) instead.

(b) Read Chapter 26.2 of Abramowitz and Stegun (*reference* book *(not a **reserve** book)* in Grainger Engineering Library), and use Equation 26.2.17 to **calculate Q(5).**
This formula is also given on Slide 30 of Lecture 26 in the Powerpoint slides.

(c) The number found in part (a) or (b) is just an *approximation* to the value of Q(5). Use the maximum error specification to find the *range* in which the actual value of Q(5) must necessarily lie. What is the *maximum relative error* in the approximation to Q(5) that you found in part (a) or (b)? Note: the relative error is defined as $\frac{|\text{true value}-\text{computed value}|}{\text{true value}}$ expressed as a percentage.

(d) On p. 972, Abramowitz and Stegun give the value of $-\log_{10}Q(5)$. Blindly trust your calculator to do the exponentiation correctly and find the *actual relative error* in the approximation to Q(5) that you found in part (a) or (b). What would the actual relative error have been if you had simply used the upper bound of Eq. (4.4) as an approximation to Q(5) as suggested by Ross? What if you had ignored Ross's suggestion and used the lower bound as an approximation to Q(5) instead?

(e) Explain why the "much easier" Equation 26.2.18 of Abramowitz and Stegun is not particularly useful for computing Q(5).

4. Let **X** denote a unit Gaussian random variable with pdf φ(u) and CDF Φ(u).

(a) What is the derivative of $\exp(-u^2/2)$ with respect to u? Use this result to find E[|**X**|].

Now, let $Q(x) = \int_x^\infty \varphi(u)\,du = \int_x^\infty (\sqrt{2\pi})^{-1}\exp\left(-\frac{u^2}{2}\right)du = 1 - \Phi(x)$.

(b) Some tables list the values of Q(x) (instead of Φ(x)) for large values of x. Why might the tabulator have chosen to specify Q(x) instead of Φ(x)? Explain briefly.
On page 211 in the 5th edition (but, regrettably, not in the 6th edition), Ross gives an upper and a lower bound on Q(x) (Eq. (4.4)). The rest of this problem leads you through a derivation of Eq. (4.4) that does not use the "obvious inequality" invoked by Ross in his proof, and it also looks at another, simpler bound.

(c) Write the integrand for Q(x) as $(\sqrt{2\pi})^{-1}u^{-1}(u\bullet\exp(-u^2/2))$ and integrate by parts to deduce that $Q(x) < x^{-1}\bullet\varphi(x)$ for $x > 0$.. Repeat the trick of re–writing and integrating by parts to show that $x^{-1}\bullet\varphi(x) - x^{-3}\bullet\varphi(x) < Q(x)$. Are these bounds useful as $x \to 0$? Why or why not? What is the asymptotic value of the ratio of the bounds as $x \to \infty$?

(d) A useful bound when x is small is $Q(x) \le (1/2)\exp(-x^2/2)$ for $x \ge 0$ in which equality holds only at x = 0. Derive this bound by first showing that $t^2 - x^2 > (t - x)^2$ for $t > x > 0$ and then applying this result to $\exp(x^2/2)Q(x) = \int_x^\infty (\sqrt{2\pi})^{-1}\exp\left[-\left(\frac{t^2 - x^2}{2}\right)\right] dt$

(e) For what values of x is this smaller than the upper bound of Eq.(4.4)?

5. A signal $x(t) = \exp(-\pi t^2)$, $-\infty < t < \infty$, is passed through a low-pass filter whose transfer function is $H(f) = \begin{cases} 2, & -1 \le f \le 1, \\ 0, & 1 < |f| > \infty. \end{cases}$ Let y(t) denote the output of the filter. Compute the value of y(0). A numerical answer is desired. [Hint: $X(f) = \exp(-\pi f^2)$, $-\infty < f < \infty$.]

6. **X** is a continuous random variable with pdf $f_{\mathbf{X}}(u) = 0.5 \exp(-|u|)$, $-\infty < u < \infty$.

(a) What is the value of $P\{\mathbf{X} \le \ln 2\}$?

(b) Find the conditional probability that $P\{|\mathbf{X}| \le \ln 2\}$ given that $\{\mathbf{X} \le \ln 2\}$.

(c) Find the numerical value of $P\{\cos(\pi\mathbf{X}/2) < 0\}$.

(d) Now suppose that **X** denotes the voltage applied to a semiconductor diode, and that the current **Y** is given by $\mathbf{Y} = e^{\mathbf{X}} - 1$. Find the pdf of **Y**.