Assigned: Wednesday, October 17, 2001
Due: Wednesday, October 24, 2001
Reading: Ross, Chapter 5
Noncredit Exercises: Ross: Chapter 5: Problems 1–8, 11, 15–19, 23, 29, 30, 31;
Theoretical Exercises: 9, 14

Problems:

1. Which of the following are valid probability density functions? Assume that the functions are zero outside the ranges specified. For those which are not valid pdfs, state at least one property of pdfs which is not satisfied. Also, state whether there exists a constant C such that C•f(u) is a valid pdf even though f(u) is not.

(a) $f(u) = |u|$ for $|u| < 1$.
(b) $f(u) = 1 - |u|$ for $|u| < 1$.
(c) $f(u) = \ln u$ for $0 < u < 1$,
(d) $f(u) = \ln u$ for $0 < u < 2$. Hint: ln u can be integrated by parts
(e) $f(u) = 2u$ for $0 < u < 1$.
(f) $f(u) = (2/3)(u - 1)$ for $0 < u < 3$.
(g) $f(u) = \exp(-2u)$, $0 < u < \infty$,
(h) $f(u) = 4\exp(-2u) - \exp(-u)$, $0 < u < \infty$.

2. **X** denotes a continuous random variable with probability density function $f_X(u)$ given by

$$f_X(u) = \begin{cases} 1 + u, & -1 < u \le 0, \\ u, & 0 < u < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find $P\{|\mathbf{X}| < 1/2\}$ and $P\{\mathbf{X} > 0 \mid \mathbf{X} < \frac{1}{2}\}$.

(b) Find the expected value of **X**.

(c) Find the expected value of $|\mathbf{X}|$.

3. The weekly demand (measured in thousands of gallons) for gasoline at a rural gas station is a random variable **X** with probability density function

$$f_X(u) = \begin{cases} 5(1-u)^4, & 0 < u < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let C (in thousands of gallons) denote the capacity of the tank (which is re-filled weekly.)

(a) If C = 0.5, (i.e., the tank holds 500 gallons) and **X** happens to have value 0.68 one particular week, (e.g. 680 people show up each wanting to purchase a gallon of gas for their snowblowers or lawnmowers), can the gas station satisfy the demand that week? That is, can the gas station supply gasoline to all those who want to buy it that week?

(b) If C = 0.5 and **X** happens to have value 0.43 some other week, can the gas station satisfy the demand during this other week? That is, can the gas station supply gasoline to all those who want to buy it that week?

(c) If C = 0.5, what is the *probability* that the weekly demand for gasoline can be satisfied? Note that if your answer is (say) 0.666…, then, in the long run, the gas station can supply the weekly demand two weeks out of three.

(d) What is the minimum value of C required to ensure that the probability that the demand exceeds the supply is no larger than 10^{-5}?

Now, suppose that the owner makes a gross profit of $0.64 for each gallon of gasoline sold. Let **Y** denote the amount of gasoline *sold* per week.

(e) How is **Y** related to **X**, the weekly *demand* for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!)

(f) What is the **average** weekly gross profit?

(g) Suppose that the owner pays $20C as weekly rent on a tank of capacity 1000C gallons. Note that $0 \le C \le 1$. (Why is a tank larger than 1000 gallons not needed?) What is the **average** weekly **net** profit and what value of C maximizes the average weekly net profit?

4. **X** is uniformly distributed on [–1, +1].

(a) If $\mathbf{Y} = \mathbf{X}^2$, what are the mean and variance of **Y**?

(b) If $\mathbf{Z} = g(\mathbf{X})$ where $g(u) = \begin{cases} u^2, & u \geq 0, \\ -u^2, & u < 0, \end{cases}$ use LOTUS (or the EZ method) to find $E[\mathbf{Z}]$

5.(a) Let $\mathbf{X}$ denote a continuous random variable whose pdf $f(u)$ is nonzero only in the finite-length interval (a, b). Show that $a < E[\mathbf{X}] < b$.

(b) Show that $\text{var}(\mathbf{X}) < (b-a)^2/4$. Hint: Let $E[\mathbf{X}] = \mu$. Then, the parabola $(u-\mu)^2$ is entirely below the straight line through the points $(a, (a-\mu)^2)$ and $(b, (b-\mu)^2)$

(c) Do these results also hold for discrete random variables whose values are in (a, b)?

(d) Show that there exists a discrete random variable $\mathbf{Y}$ taking on values in $[a, b]$ such that $\text{var}(\mathbf{Y}) = (b-a)^2/4$. Note that equality can hold now.