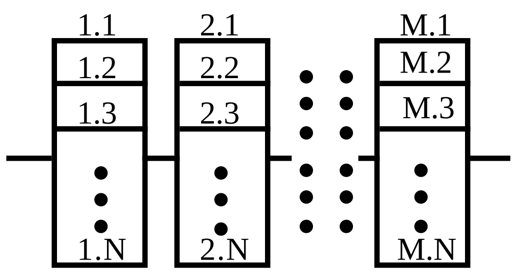
**University of Illinois** | **Problem Set #8** | **ECE 313**

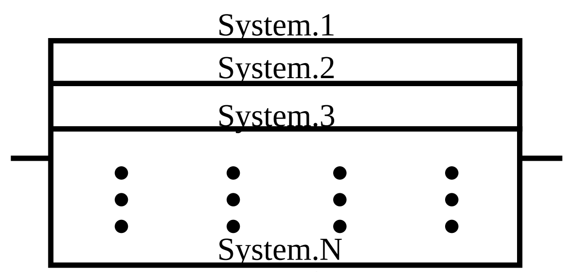
**Fall 2001**

**jAssigned:** Wednesday, October 10
**Due:** Wednesday, October 17
**Reading:** Ross, Chapter 4, Sections 1, 2 and 9; and Chapter 5
**Noncredit Exercises:** Chapter 5: Problems 1-8; Theoretical Exercises: 1, 8
**Problems:**

**1.** ["… It's not your father's Oldsmobile…"] A system works if and only if all of its M components (numbered 1 through M) work. Each component fails (independently) with probability p. Consider two possible means of obtaining a more reliable system. We can replicate each component N times as shown in the graph model on the left. Or, we can replicate the entire system N times as shown in the graph model on the right. In either case, the result is called a replicated system. Note that both methods use the *same* number N of each component .



N-ary replication of each component



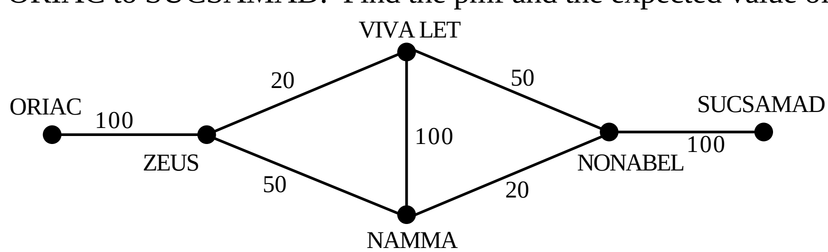
N-ary replication of entire system

A concrete example of the question we wish to consider is:
Which of the following two methods provides more reliable transportation?

- a single gigantic car with N engines, N transmissions, N brakes, … etc. that works (i.e. provides us with transportation) as long as at least one of its engines and at least one of its transmissions, and at least one of its brakes … works
- N separate ordinary cars that fail as soon as any one of their components fail, but which together provide us with transportation as long as at least one car is in working condition.

**(a)** For each model, find the probability of replicated system failure in terms of p, N and M.
**(b)** Suppose that M = 5 and p = 0.2. If it is desired that the replicated system failure probability be less than 0.001, what should N be in each case?
**(c)** Repeat part (b) assuming that there are M subsystems numbered I, II, III, IV, … , M. Hint about the value of M: This homework is due on October XVII, MMI.

**2.** ["Reach out and touch someone"] MiddleEast Bell, a division of Horizon Corp., has built a telephone network as shown below. Terrorists attack each of the seven links. The attacks may be considered to be independent events, and the attack on a link succeeds in severing the link with probability p. If a link is severed, switches automatically re-route calls so as to avoid the failed link (if possible).
**(a)** What is the probability of being able to call from ORIAC to SUCSAMAD?
**(b)** Given that it is possible to call from ORIAC to SUCSAMAD, what is the conditional probability that the ZEUS to NAMMA link is in working condition?
**(c)** The link capacities (i.e., the numbers of telephone calls that the links can carry (in either direction)) are as marked on the diagram. Let **X** denote the number of calls that can be made from ORIAC to SUCSAMAD. Find the pmf and the expected value of **X**.



**3.** Two teams A and B play in the World Series which consists of a series of baseball games (each ending in a win for one of the teams) that continues until one team has won four games. Thus, the number of games in the Series is 4 or 5 or 6 or 7. Let the random variable $\mathbf{X}$ denote the number of games played in the Series. Assume that the outcomes of the various games are independent.

**(a)** Assuming that each team is equally likely to win each game, calculate the pmf of $\mathbf{X}$, and the expected value of $\mathbf{X}$. [Hint: it is not a binomial pmf]. Which team is more likely to win the World Series? Which team is more likely to win the World Series in five games?

**(b)** Now suppose that a team playing in its own stadium in front of its own fans has probability 0.55 of winning the game, and that Team A has the overall *home field advantage*, that is, it gets to play more games at home than Team B if the Series extends to seven games. Note also that A gets to play the seventh and deciding game at home if the series extends that far. Two common formats for arranging the games are for A and B each to play two games at home, and then to alternate between cities as needed (AABBABA) or for A to play the first two games and the sixth and seventh( if necessary) at home (AABBBAA). For each format, find the pmf of $\mathbf{X}$ and the probability that A wins the Series. Explain why the answers would be the same if the first four games were arranged ABAB instead of AABB. Then explain why major league baseball does not use the ABABABA format.

**(c)** Compare the answers to parts (a) and (b) to determine how much the overall *home field advantage* is worth. Which format gives A the best chance of winning?

**(d)** You do not know the results of the first four games. Given that you know *only* that $\mathbf{X} = 5$, which hypothesis (“A won the Series” or “B won the Series”) has greater *a posteriori* probability? The answer depends on the format AABBABA or AABBBAA, so answer the problem for both formats.

**4.** The number of $\alpha$-particles emitted by a source during a unit time interval can be modeled as a Poisson random variable $\mathbf{X}$ with parameter $\lambda$. Let $A_n$ denote the event $\{\mathbf{X} = n\}$.

**(a)** (This one is a gimme) What is $P\{A_n\}$?

The $\alpha$-particles are detected by means of a (imperfect) Geiger counter which detects a particle with probability $p < 1$. The detections of the various particles can be considered to be independent events. Thus, if event $A_n$ has occurred, that is, if n particles have been emitted, the Geiger counter reading can be modeled as a binomial random variable $\mathbf{Y}$ with parameters (n,p). In short, $p_{\mathbf{Y}|A_n}(k|A_n)$, the *conditional* pmf of $\mathbf{Y}$ given the event $A_n$, is a binomial pmf: $p_{\mathbf{Y}|A_n}(k|A_n) = \binom{n}{k}p^k(1-p)^{n-k}$ for $0 \le k \le n$..

**(b)** What is the conditional mean of $\mathbf{Y}$ given the event $A_n$? From this, find the unconditional mean of $\mathbf{Y}$.

**(c)** What is the unconditional pmf of $\mathbf{Y}$? [Hint: $\mathbf{Y}$ takes on *all* nonnegative integer values…]

**(d)** Let $B_k$ denote the event $\{\mathbf{Y} = k\}$. What is the conditional pmf of $\mathbf{X}$ given $B_k$?

**(e)** What is the conditional mean of $\mathbf{X}$ given $B_k$?

**(f)** What is $P\{B_k\}$? Use this result together with that of part (d) to verify that the theorem of total probability gives us that the unconditional pmf of $\mathbf{X}$ is Poisson with parameter $\lambda$.

**(g)** We can observe the Geiger counter reading $\mathbf{Y}$, but we wish to know the value of $\mathbf{X}$? If the Geiger counter reading is k, i.e. the event $\{\mathbf{Y} = k\}$ is observed, what is the maximum-likelihood estimate of $\mathbf{X}$?

**5.** The random variable $\mathbf{X}$ has the CDF shown in Figure 4.1 on p. 133 of Ross (5th edition) or Figure 4.7 on p. 168 of Ross (6th edition). Find $E[\mathbf{X}]$.

**6.** The random variable $\mathbf{X}$ has probability density function

$$f_{\mathbf{X}}(u) = \begin{cases} \alpha(1-u), & 0 < u < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

**(a)** Find $P\{6\mathbf{X}^2 > 5\mathbf{X} - 1\}$.
**(b)** Find $F_{\mathbf{X}}(u)$. Be sure to specify the value of $F_{\mathbf{X}}(u)$ for all u.

**Noncredit optional exercise:**

Some of you were concerned about the use of the Chebyshev inequality in Problem 2 of Problem Set #5 to get that at least 4123 repetitions were needed to achieve the desired error probability. Here is a tighter result. With $\mathbf{X}$ denoting an (n,p) binomial random variable,

**(a)** Show that if $\lambda > 0$, then $\exp\left[\lambda\left(u - \frac{n}{2}\right)\right] > 1$ for all $u > \frac{n}{2}$ and that therefore

$$P\{\mathbf{X} > \tfrac{n}{2}\} = \sum_{u>n/2}^{n} p_{\mathbf{X}}(u) < \sum_{u>n/2}^{n} \exp\left[\lambda\left(u - \frac{n}{2}\right)\right] p_{\mathbf{X}}(u) < \sum_{u=0}^{n} \exp\left[\lambda\left(u - \frac{n}{2}\right)\right] p_{\mathbf{X}}(u)$$

i.e., $P\{\mathbf{X} > \frac{n}{2}\} < E\left[\exp\left(\lambda\left(\mathbf{X} - \frac{n}{2}\right)\right)\right]$.

This result is called a Chernoff bound (see also Chapter 8 of Ross).

**(b)** Use LOTUS to prove directly from the known pmf of $\mathbf{X}$ that

$$E[\exp(\lambda\mathbf{X})] = (1 - p + p\bullet\exp(\lambda))^n$$

and hence that $P\{\mathbf{X} > n/2\} < \exp(-n\lambda/2)\bullet(1 - p + p\bullet\exp(\lambda))^n$ for all values of $\lambda > 0$.

**(c)** Evaluate the above bound on $P\{\mathbf{X} > n/2\}$ at $\lambda = \ln 2$ and at $\lambda = -\ln p$.

**(d)** Prove that as a function of $\lambda$, $\exp(-n\lambda/2)\bullet(1 - p + p\bullet\exp(\lambda))^n$ has minimum value $\left(2\sqrt{p(1-p)}\right)^n$ at $\lambda = \ln((1-p)/p)$. How different is $\ln((1-p)/p)$ from $-\ln p$?

**(e)** Repeat part (c) of Problem #2 of Problem Set #5 to choose the value of n using the Chernoff bound rather than the Chebyshev inequality.
Who gets the raise for designing an efficient communication system, you who slogged through to here or your pal who stopped at part (c) of Problem #2 of Problem Set #5?