University of Illinois — **Problem Set #4** — **ECE 313**

Fall 2001

Assigned: Wednesday, September 12
Due: Wednesday, September 19
Reading: Ross, Chapter 4.1, 4.3–4.5, 4 .7, and Chapter 3
Noncredit Exercises: Chapter 4, Problems: 34, 35-38, 43, 48; Theoretical Exercises: 9, 13, 15
Problems:

1. The prospectus of GoGoDotCom Inc., an investment management service, states that their goal is to double the value of their clients' investments in a week via day trading of Internet stocks. (The Securities and Exchange Commission insists, as usual, that a disclaimer be included that there is no guarantee that the goal will be met.) The TV commercials proclaim "On average, our clients triple their money in five weeks!" You decide to invest \$32 (hey, you are a student on a tight budget) with GoGoDotCom Inc. for a period of five weeks. Let **X** denote the value (in dollars) of your investment at the end of this period.
Now suppose that GoGoDotCom Inc.has a 50% chance of doubling your investment and 50% chance of losing half your investment. That is, if you invest \$C with them, then, a week later, your investment is equally likely to be worth \$2C or \$C/2. Assume that each week's performance is an independent trial that doubles or halves the value that your investment had at the beginning of the week.

(a) What are the possible values of **X**?
(b) What is the pmf of the random variable **X**?
(c) What is the expected value of **X**? Is the TV commercial an accurate statement?
(d) What is the probability that you will lose money on this investment? i.e. find P{**X** < 32}.
(e) An investment of \$32,000 with GoGoDotCom Inc. would be worth \$1000**X** in five weeks. Assuming that you have the money, would you be willing to make such an investment? Why or why not? Would you be willing to borrow the money from your parents to make the investment? How about borrowing the money from a loan shark?

2. Let **X** denote a binomial random variable with parameters (N, p). What is the probability that **X** is an even integer? Remember that 0 is an even integer.
[Hint: What is $(x+y)^N + (x - y)^N$?]

3. There are N multiple-choice questions on a certain examination. A student knows the answer to K of these and marks the answer sheet accordingly. For the remaining N – K questions, the student guesses randomly among the five choices. The examiner can easily determine C, the number of correct answers on the answer sheet, but is more interested in estimating the value of K, since K is a better measure of the student's knowledge than C. (Educators like to nitpick about such subtle differences!). Note that the number of **w**rong answers can be modeled as a binomial random variable **W** with parameters (N – K, 0.8).

(a) The examiner notes that n questions have been answered incorrectly by the student, i.e. the event {**W** = n} is observed. Write an expression for P{**W** = n} in terms of N, K, and n.
(b) Obviously, $0 \le K \le N-n$. Now, use the method used in the proof of Prop. 7.1, Chapter 4 of Ross to show that of all possible assumptions K = 0, K = 1, K = 2, … , K = N–n that the examiner might make, the assumption that K is the largest integer not exceeding N – 1.25n + 1, i.e. estimating K as $\hat{K} = \lfloor N - 1.25n + 1 \rfloor$ maximizes P{**W** = n}. $\hat{K}$ is the called the maximum-likelihood estimate of K. Find the numerical value of $\hat{K}$ for the case N = 100 and n = 8.
(c) Since C = N – n, examiners generally subtract one-fourth of the wrong answers from C and estimate the value of K as $\tilde{K} = C - \lfloor 0.25n \rfloor \approx N - 1.25n$. This is called applying the guessing penalty, and it *can* hurt scores slightly in the sense that the examiner's estimate $\tilde{K}$ *might* be smaller than the maximum-likelihood estimate $\hat{K}$ found in part (b).
Compare $\hat{K}$ and $\tilde{K}$ for the case N = 100 and n = 8.
Compare $\hat{K}$ and $\tilde{K}$ for the case N = 100 and n = 10.
(d) If N = 100 and K = 90, which of the events {**W** = 0}, {**W** = 1}, … , {**W** = 10} has the largest probability? (Hint: See Proposition 7.1, p. 150 of Ross). Suppose that this largest

probability event actually occurred. Does the examiner's estimate $\tilde{K}$ correctly estimate K? Does the maximum-likelihood estimate $\hat{K}$ correctly estimate K?

(e) Continuing to assume that N = 100 and K = 90, what happens if by sheer dumb luck the student manages to guess right on 6 problems so that the event {**W** = 4} occurs? Compare this to the case when the event {**W** = 10} occurs (as in part (c)).

(f) Continuing to assume that N = 100 and K = 90, find the probabilities that the examiner's estimate $\tilde{K}$ respectively overestimates, underestimates, and correctly estimates K.

(g) **Optional Noncredit Exercise:** Suppose that for each of the 10 questions to which the answer is not known to the student, the student can nonetheless correctly eliminate three answers as being obviously wrong. The student then chooses at random between the other two answers. Which of the events {**W** = 0}, {**W** = 1}, ... , {**W** = 10} is the most probable? If this is the event which actually occurs, what is the examiner's estimate $\hat{K}$? (Note that this, in essence, gives *some* "partial credit" by rewarding the student for the partial knowledge that three of the five answers to each problem are wrong.)

(g) **Optional Noncredit Exercise:** Write a 500-word essay on why it is more important to be lucky than smart.

4. In the game of Chuck-A-Luck often played in MidWestern carnivals and fairs, a player bets money (the stake) on one of the numbers 1, 2, 3, 4, 5, 6. Three fair dice are rolled (so that the sample space has $6^3 = 216$ equally likely outcomes in it). If the number chosen shows up on one, two, or all three of the dice, the player ***wins*** respectively one, two, or three times the money that was staked; and, of course, the original money staked is also returned to the player. If the chosen number shows up on none of the dice, the player ***loses*** the money that was staked. **X** is the random variable that denotes the amount of money won for a $6 bet.

(a) What are the possible values that **X** can take on? Remember that negative values of **X** denote losses.

(b) What is the probability mass function (pmf) of **X**?

(c) What is the expected value of **X**?

(d) ["Always go out a winner"] A player splits his $6 bet into a $1 bet on each of the six numbers with the idea that at least one, and possibly as many as three, of his bets will be winners. Let **Y** denote his winnings. What are the values taken on by **Y**? What is the expected value of **Y**? Compare your answer to the expected value of **X** found in part (c).

5. Ross, Problem 4, p. 173 (5th edition) or p. 171 (6th edition)

6. **Try using a spreadsheet or MATLAB™ or Mathematica™ for this problem.** ["Extra! Extra! Read all about it!"] A newsboy purchases 50 newspapers for 35 cents each from the publisher and sells them for 60 cents each. He can recycle any unsold papers and get 25 cents back for each. A total of 100 people pass by him each day, and each decides (independently of all the others) to ask to buy a paper with probability 0.6. Thus, the daily demand **X** for papers can be modeled as a binomial random variable with parameters (100,0.6).

(a) What is the probability that the newsboy sells all 50 newsapers that he bought?

(b) Let **Z** denote the daily profit (in cents) that the newsboy makes. What is his maximum profit? What is his maximum loss? More generally, express **Z** in terms of **X**.

(c) Compute the newsboy's **average** daily profit.

(d) The newsboy has been buying 50 papers for some months when he realizes that on many days he is turning away customers — he could sell more papers than 50 if he only had bought them. As a prudent businessman, he decides to test the waters by buying one extra paper the next day.
What is the probability that he can sell this extra paper?
What is the average **additional** profit he makes from this extra paper?

(e) Repeat part (d) for the day after when he daringly buys 52 papers. Note that you are being asked to compute his average additional profit from the 52nd paper (over and above the profit from 51 papers that he bought the previous day.) Is the average **additional** profit from the 52nd paper larger or smaller than the average **additional** profit from the 51st paper? (This is called the law of diminishing returns.)

(f) Let A(N) denote the average **additional** profit from the (N+1)th paper over the profit from N papers. Thus, your answer to part (d) is A(50) and the answer to part (e) is A(51). Show that $A(N) = 25 - 35 \cdot P\{\mathbf{X} \le N\}$. Show that this formula implies that A(N) is negative for large values of N (Hint: what is $P\{\mathbf{X} \le 100\}$?), that is, the newsboy loses money if he buys too many extra papers. (See? he is cautious with good reason!)

(g) How many papers should he purchase to maximize his average profit?

7.(a) Use linearity of the expectation operator to prove that $\text{var}(\mathbf{X}) = E[\mathbf{X}(\mathbf{X}-1)] - E[\mathbf{X}]E[\mathbf{X}-1]$.

(b) Let $\mathbf{X}$ denote a discrete random variable with minimum and maximum values a and b respectively, where $a < b$. Prove that the Lake Woebegon Inequality: $a < E[\mathbf{X}] < b$.