Assigned: Wednesday, August 29
Due: Wednesday, September 5
Reading: Ross, Chapter 1 and Chapter 2, Sections 1 to 5.
Noncredit Exercises: (Do not turn these in)

	Problems	*Theoretical Exercises*	*Self-Test Problems*
Chapter 1	1–5, 7, 9	4, 8, 13	1-15
Chapter 2	3-5, 8-14, 23, 27-29, 36, 38, 39, 41, 43, 45	1–4, 5, 7, 10, 12, 16, 18, 19, 20	1-8

Problems:

1. This problem will help you recollect some fundamental mathematical results that will be used over and over again in this course.

(a) Write down the first three terms, and an expression for the general term (say, the k-th), of the *Taylor series* for a differentiable function f(x) in the neighborhood of x = 0.

(b) Now suppose f(x) is a polynomial of degree n > 0. What is the n-th derivative of f(x)? What is the (n+1)-th derivative of f(x)? What is the (n+2)-th derivative of f(x)?

(c) Henceforth, let $f(x) = (1+x)^n$, $n > 0$.
State True or False: If we multiply out the terms, we will get a polynomial of degree n.

(d) It is. of course, not necessary to multiply out all the (1+x) terms in f(x) in order to find its derivative. Show that this is true by finding the first and second derivatives of $(1+x)^n$. Now use the result of part (a) to find the first three terms of the Taylor series for $(1+x)^n$.

(e) For $0 \le k \le n$, what is the k-th derivative of $(1+x)^n$, and what is the k-th term of the Taylor series for $(1+x)^n$?

(f) Repeat part (e) for $k > n$.

(g) State True or False: The Taylor series for $(1+x)^n$ contains terms of degree greater than n.

(h) Congratulations! You have just re-discovered the binomial theorem for positive exponents. Now suppose that $f(x) = (1-x)^{-n} = \frac{1}{(1-x)^n}$, where, as before, we assume that $n > 0$.
Repeat part (d) to find the first three terms of the Taylor series for $(1-x)^{-n}$.

(i) Does the Taylor series for $(1-x)^{-n}$ contain terms of degree greater than n? If so, what is the (n+1)-th term in the Taylor series? If not, what is the n-th term in the Taylor series?

(j) Use the results of parts (h) and (i) to write down the Taylor series for $(1-x)^{-1}$ and $(1-x)^{-2}$. **These two results and the one in parts (e)-(g) are used so often in the course that it is best to memorize them!**

(k) I hope that you found that $(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots$ which gives the curious result that $-1 = \infty$ on setting x = 2 on both sides. Is this correct? If not, what is wrong?

2. We noted that the binomial coefficient $\binom{n}{k}$ is the coefficient of x^k in the Taylor series (or binomial theorem) expansion of $(1+x)^n$, and it also denotes the number of subsets of size k from a set of n objects.

(a) Use the binomial theorem to expand $(1-x)^n$. DO NOT use summation (Σ) notation: I want to see the first four terms explicitly listed, and I also want to know the coefficient of x^n.

(b) Compare the result of part (a) to the expansion of $(1+x)^n$. Some powers of x have positive signs in both expansions. Which powers are these? Which powers have opposite signs?

(c) Show that the Taylor series for $(1+x)^n + (1-x)^n$ has even powers of x only.

(d) Set x = 1 in the result of part (c) and thus show that the total number of subsets of size k, where *k is even* , is 2^{n-1}. (Note: zero is an even number)

(e) Explain why the result of part (d) implies that the total number of subsets of size k, where *k is odd* , is also 2^{n-1}.

3. Ross, Problem 6 of Chapter 2. "Let E, F, G, be three events…"

4.(a) Consider choosing a 3-member committee at random from a group of 6 men and 6 women. Find the number of possible committees that consist of 2 men and 1 woman. Argue by symmetry that an equal number of committees consist of 2 women and 1 man.

(b) An ice-cream manufacturer makes unflavored ice-cream and then creates "specialty flavors" by blending one or more of the five essences: vanilla, chocolate, fudge, mint, and almond. Thus, vanilla almond fudge ice-cream is created by blending in the three essences vanilla, almond, and fudge into the unflavored ice-cream. How many specialty flavors can the manufacturer create? **Optional non-credit exercise:** Identify the manufacturer!

5. Let $\Omega = \{x_1, x_2, \ldots x_n\}$ denote a finite sample space of n outcomes with probabilities $p_1, p_2, \ldots p_n$ where $p_1 + p_2 + \ldots + p_n = 1$. There are 2^n events defined on this sample space. With any event A, we can associate an n-bit word **a** where for $1 \le i \le n$, the i-th bit of **a** is 1 if and only if x_i A. We write $A \leftrightarrow \mathbf{a}$ to denote this association (one-one map).

(a) If $A \leftrightarrow \mathbf{a}$ and $B \leftrightarrow \mathbf{b}$, express the n-bit words associated with A^c, A B and $A \cap B$ in terrms of **a** and **b**.

(b) Express P(A) in terms of **a** and $\mathbf{p} = (p_1, p_2, \ldots p_n)$.

(c) Find the "average probability" of an event by adding up the probabilities of all 2^n events and dividing the resulting sum by 2^n.

(d) Suppose that the outcome x_3 occurred on a trial. Then, 2^{n-1} events occurred, and 2^{n-1} did not. What can be said about the n-bit words associated with the events that did occur? What is the "average probability" of the 2^{n-1} events that occurred? Note that now you have to add up the probabilities of 2^{n-1}specific events and divide the resulting sum by 2^{n-1}.

If you cannot do parts (c) and (d) in general, you will get 50% credit for doing them for $\Omega = \{x_1, x_2, x_3, x_4\}$ and $P(x_1) = 1/2$, $P(x_2) = 1/4$, $P(x_3) = P(x_4) = 1/8$. You will need to find (and add) the probabilities of many events, so work systematically.

6. An experiment consists of observing the contents of an eight-bit shift register. Assume that all $2^8 = 256$ bytes are equally likely to be the contents of the shift register.

(a) Let A denote the event that the least significant bit in the shift register is a 1. What is P(A)?

(b) Let B denote the event that the shift register contains 4 0's and 4 1's. What is P(B)?

(c) What is P(A B) ? What is $P(A \cap B)$? What is the probability that exactly one of the events A and B occurs, i.e. what is P(A B)?

7. The manufacturer of a cereal tests samples from the production line to see if the samples snap, crackle, and pop as advertised. Let A, B, and C denote respectively the events that the sample **does not** snap, **does not** crackle, and **does not** pop. The manufacturer's tests show that P(A) = P(C) = 0.3, P(B) = 0.2, P(AB BC AC) = 0.3, P(ABC) = 0.05, P(AB) = 0.1, and P(AC) = 2P(BC).

(a) Sketch the sample space and indicate on it the events A, B, and C.

(b) What is the probability that the cereal snaps, crackles, and pops ?

(c) Cereal that fails exactly one test is sold to discount supermarket chains at lower prices to be marketed under the brand names Soggies, Blecchies, and Mushies. What is the probability of the sample failing the snap test **only**? the crackle test **only**? the pop test **only**?