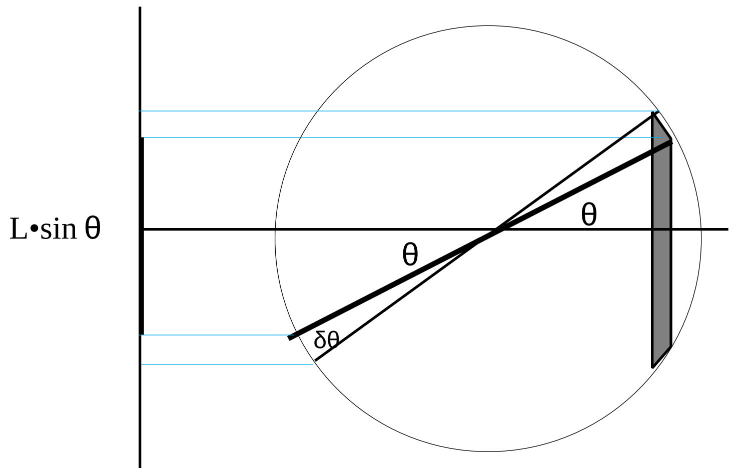
1.(a) Let **X** denote the length of the shadow. Then, $0 \le \mathbf{X} \le L$. Also, as can be seen from the diagram below, if the needle makes an angle of at least θ and no more than θ + δθ to the normal to the plane, then the shadow is of length between $L \cdot \sin\theta$ and $L \cdot \sin(\theta + \delta\theta)$



If the angle is between θ and θ + δθ, the tip of the needle must lie in a circular strip (shaded) on the sphere surface. The strip is of radius $(L/2) \cdot \sin\theta$ and width $(L/2) \cdot \delta\theta$. Hence, its area is $[2\pi(L/2) \cdot \sin\theta] \cdot (L/2) \cdot \delta\theta = (\pi L^2/2) \cdot \sin\theta\, \delta\theta$ and the probability that the point of the needle lies in that strip is just the area divided by the surface area of the hemisphere which is $2\pi(L/2)^2 = \pi L^2/2$, i.e. the probability is just $\sin\theta\, \delta\theta$. Thus, for any $\theta_0 \le \pi/2$, $P\{\mathbf{X} \le L \cdot \sin\theta_0\} = \int_0^{\theta_0} \sin\theta\, d\theta = 1 - \cos\theta_0 = 1 - \sqrt{1 - \sin^2\theta_0}$. Set $L \cdot \sin\theta_0 = x$.

Then, we have $F_X(x) = 1 - \sqrt{1 - (x/L)^2}$, $0 \le x \le L$. Hence,

$$f_X(x) = \begin{cases} \dfrac{x}{L\sqrt{L^2 - x^2}}, & 0 \le x \le L, \\ 0, & \text{otherwise.} \end{cases}$$

(b) $E[\mathbf{X}] = \int_0^L P\{\mathbf{X} > x\}\, dx = \int_0^L \sqrt{1 - (x/L)^2}\, dx = \int_0^{\pi/2} L \cos^2\theta\, d\theta$ on substituting $x = L \cdot \sin\theta$.

This is a standard integral **that should be familiar to all electrical engineering students.**

$\int_0^{2\pi} \cos^2\theta\, d\theta = \pi$ and $\int_0^{\pi/2} \cos^2\theta\, d\theta = \pi/4$ since the average power in a unit-amplitude sinusoid is 1/2.
Hence, $E[\mathbf{X}] = L \cdot (\pi/4)$.

2.(a) The lifetime is an exponential random variable with parameter λ. Hence, the *average* lifetime is $\lambda^{-1} = (-\ln 0.999)^{-1} \approx 999.5$ weeks which is more than 19 years! The *median* lifetime is T where $P\{\mathbf{X} > T\} = \exp(-\lambda T) = 1/2$, i.e. $T = \lambda^{-1} \ln 2 \approx 692.8$ weeks which is only 13 years and a few months.

(b) $P\{\mathbf{X} > 1\} = \exp(-\lambda) = 0.999$ so even with this long-lived module, we can only be 99.9% sure that the thing is going to last for a week!

(c) $\{\mathbf{Y} > t\}$ is the (disjoint) union of the four events $\{\mathbf{X}_1 > t\} \cap \{\mathbf{X}_2 > t\} \cap \{\mathbf{X}_3 > t\}$, $\{\mathbf{X}_1 > t\} \cap \{\mathbf{X}_2 > t\} \cap \{\mathbf{X}_3 \le t\}$, $\{\mathbf{X}_1 > t\} \cap \{\mathbf{X}_2 \le t\} \cap \{\mathbf{X}_3 > t\}$, and $\{\mathbf{X}_1 \le t\} \cap \{\mathbf{X}_2 > t\} \cap \{\mathbf{X}_3 > t\}$.

(d) Since $P\{\mathbf{X}_i > t\} = \exp(-\lambda t)$, we readily find that the probability of the first event in (c) is $\exp(-3\lambda t)$, while the other three have probability $\exp(-2\lambda t)\,[1 - \exp(-\lambda t)]$. Hence, $P\{\mathbf{Y} > t\} = 3\exp(-2\lambda t) - 2\exp(-3\lambda t)$.

Furthermore, $E[\mathbf{Y}] = \int_0^\infty P\{\mathbf{Y} > t\}\, dt = \int_0^\infty 3\exp(-2\lambda t) - 2\exp(-3\lambda t)\, dt = (3/2\lambda) - (2/3\lambda) = (5/6\lambda) < 1/\lambda$

so that the average lifetime of the TMR system is actually smaller than that of the individual modules! To find the median lifetime, we set $P\{\mathbf{Y} > T\} = 3\exp(-2\lambda T) - 2\exp(-3\lambda T) = 3p^2 - 2p^3 = 1/2$. It is easy to show that p = 1/2 is one solution while the other two solutions correspond to p < 0 and p > 1. Hence, the median lifetime of the TMR system is also λ^{-1} ln 2 ≈ 692.8 weeks. Thus, in terms of these averages, the TMR system is just a waste of money; you have to pay more than three times as much (the majority-logic gate, not being one of the best things in life, is not free!) and the performance seems to be worse than that of a single module.

(e) $P\{\mathbf{Y} > 1\} = 3\exp(-2\lambda) - 2\exp(-3\lambda) = 3(0.999)^2 - 2(0.999)^3 = 0.999997002$ which is *much better* than the comparable figure for the single module.

(f) $P\{\mathbf{Y} > t\} = 0.999$ for $t = -\lambda^{-1}\ln 0.98163 \approx 18.53$..weeks which is more than 4 months! Thus, we can be 99.9% sure that the TMR system will work for at least 18.53 weeks. This is **much** longer than the week that a single module might be expected to last with 99.9% probability. It is in **these** matters that the TMR system makes a difference, and why it is one of the choices available to the system designer who wants to create reliable systems. As noted above, the average lifetime of the TMR system is worse than that of the single module. This is an illustration of the truism that complex systems have worse reliability than simple systems. What the TMR system is giving up is average lifetime in return for **much** higher reliability in early life.

3.(a) $E[\mathbf{Z}] = 2\{E[\mathbf{X}^2] - E[\mathbf{Y}^2]\} = 2\{\text{var}(\mathbf{X}) - \text{var}(\mathbf{Y}) + E^2[\mathbf{X}] - E^2[\mathbf{Y}]\} = -40$.

(b) $\text{cov}(\mathbf{T},\mathbf{U}) = \text{cov}(2\mathbf{X}+\mathbf{Y}, 2\mathbf{X}-\mathbf{Y}) = 4\bullet\text{var}(\mathbf{X}) - \text{var}(\mathbf{Y}) = 7$.

(c) $E[\mathbf{W}] = E[3\mathbf{X}+\mathbf{Y}+2] = 3E[\mathbf{X}] + E[\mathbf{Y}] + 2 = 9$.
$\text{var}(\mathbf{W}) = \text{var}(3\mathbf{X}+\mathbf{Y}+2) = \text{var}(3\mathbf{X}+\mathbf{Y})$ (why?) $= 9\bullet\text{var}(\mathbf{X})+\text{var}(\mathbf{Y})+6\bullet\text{cov}(\mathbf{X},\mathbf{Y}) = 9\bullet4+9+6\bullet(0.1\bullet2\bullet3) = 48.6$.

(d) $P\{\mathbf{W} > 0\} = 1 - \Phi((0-9/\sqrt{48.6}) = 1 - \Phi(-9/\sqrt{48.6}) = \Phi(9/\sqrt{48.6})$ (why?)

4.(a) $\text{var}(\mathbf{X}+\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) + 2\bullet\text{cov}(\mathbf{X},\mathbf{Y}) = 36$
$\text{var}(\mathbf{X}-\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) - 2\bullet\text{cov}(\mathbf{X},\mathbf{Y}) = 64$. Hence, $\text{cov}(\mathbf{X},\mathbf{Y}) = -7$.
If $\text{var}(\mathbf{X}) = 3\bullet\text{var}(\mathbf{Y})$, then $\text{var}(\mathbf{X}+\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) + 2\bullet\text{cov}(\mathbf{X},\mathbf{Y}) = 4\bullet\text{var}(\mathbf{Y}) - 14 = 36$, so that $\text{var}(\mathbf{Y}) = 12.5$, $\text{var}(\mathbf{X}) = 37.5$, and $\rho = \text{cov}(\mathbf{X},\mathbf{Y})/\sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})} = -7/(12.5\sqrt{3})$.

(b) If $\text{var}(\mathbf{X}+\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) + 2\bullet\text{cov}(\mathbf{X},\mathbf{Y}) = \text{var}(\mathbf{X}-\mathbf{Y}) = \text{var}(\mathbf{X}) + \text{var}(\mathbf{Y}) - 2\bullet\text{cov}(\mathbf{X},\mathbf{Y})$, then it must be that $\text{cov}(\mathbf{X},\mathbf{Y}) = 0$ and hence **X** and **Y** are uncorrelated.

(c) The values of var(**X**) and var(**Y**) do not affect the covariance computation at all. So, the random variables may or may not be uncorrelated.

5. It is easy to see that $f_{\mathbf{X}}(u)$ has constant value 2/3 for 0 < u < 1/2, and constant value 4/3 for 1/2 < v < 1.

(a) Hence, $E[\mathbf{X}] = \int_0^{1/2} u\bullet2/3\ du + \int_{1/2}^{a} u\bullet4/3\ du = \frac{(1/2)^2}{3} - \frac{0^2}{3} + \frac{2\bullet1^2}{3} - \frac{2\bullet(1/2)^2}{3} = \frac{7}{12}$

and $E[\mathbf{X}^2] = \int_0^{1/2} u^2\bullet2/3\ du + \int_{1/2}^{a} u^2\bullet4/3\ du = \frac{2\bullet(1/2)^3}{9} - \frac{2\bullet0^3}{9} + \frac{4\bullet1^3}{9} - \frac{4\bullet(1/2)^3}{9} = \frac{5}{12}$.

Therefore, $\text{var}(\mathbf{X}) = E[\mathbf{X}^2] - (E[\mathbf{X}])^2 = \frac{5}{12} - \frac{49}{144} = \frac{11}{144}$.

(b) Since the pdf of **Y** is the same as the pdf of **X**, it has the same mean and variance.

(c) $E[\mathbf{XY}] = \int_{u=0}^{1}\int_{v=0}^{1} uv\bullet4/3\ dv\ du - \int_{u=0}^{12}\int_{v=0}^{1/2} uv\bullet4/3\ dv\ du = \frac{4}{3}\left[\frac{1}{2}\times\frac{1}{2} - \frac{1}{8}\times\frac{1}{8}\right] = \frac{5}{16}$ (whoa! why?)

Hence, $\text{cov}(\mathbf{X},\mathbf{Y}) = E[\mathbf{XY}] - E[\mathbf{X}]E[\mathbf{Y}] = \frac{5}{16} - \frac{49}{144} = \frac{-4}{144} = \frac{-1}{36}$ and thus $\rho = \frac{\text{cov}(\mathbf{X},\mathbf{Y})}{\sqrt{\text{var}(\mathbf{X})\text{var}(\mathbf{Y})}} = \frac{-4}{11}$.

(d) $E[\min\{\mathbf{X},\mathbf{Y}\}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \min(u,v) f_{\mathbf{X},\mathbf{Y}}(u,v)\ du\ dv = \int_{u<v}\int u f_{\mathbf{X},\mathbf{Y}}(u,v)\ du\ dv + \int_{v<u}\int v f_{\mathbf{X},\mathbf{Y}}(u,v)\ du\ dv$

$= \int_{v=1/2}^{1}\int_{u=0}^{v} u\bullet4/3\ du\ dv + \int_{u=1/2}^{1}\int_{v=0}^{u} v\bullet4/3\ dv\ du = 2\bullet4/3 \int_{v=1/2}^{1}\int_{u=0}^{v} u\ du\ dv = 8/3\int_{v=1/2}^{1} v^2/2\ dv = \frac{7}{18}$.

(e) $E[\max\{\mathbf{X},\mathbf{Y}\}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \max(u,v) f_{\mathbf{X},\mathbf{Y}}(u,v)\ du\ dv = \int_{u<v}\int v f_{\mathbf{X},\mathbf{Y}}(u,v)\ du\ dv + \int_{v<u}\int u f_{\mathbf{X},\mathbf{Y}}(u,v)\ du\ dv$

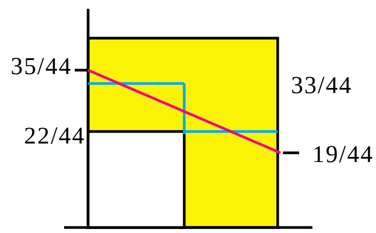
$= \int_{v=1/2}^{1}\int_{u=0}^{v} v\bullet4/3\ du\ dv + \int_{u=1/2}^{1}\int_{v=0}^{u} u\bullet4/3\ dv\ du = 2\bullet4/3 \int_{v=1/2}^{1}\int_{u=0}^{v} v\ du\ dv = 8/3\int_{v=1/2}^{1} v^2\ dv = \frac{7}{9}$.

(f) Yes, $E[\max\{\mathbf{X},\mathbf{Y}\}] = \frac{7}{9} > E[\min\{\mathbf{X},\mathbf{Y}\}] = \frac{7}{18}$ as indeed it should be for any random variables. Since $\max\{\mathbf{X},\mathbf{Y}\} \geq \min\{\mathbf{X},\mathbf{Y}\}$, $E[\max\{\mathbf{X},\mathbf{Y}\}]$ cannot be smaller than $E[\min\{\mathbf{X},\mathbf{Y}\}]$.

(g) $E[\min\{\mathbf{X},\mathbf{Y}\}] + E[\max\{\mathbf{X},\mathbf{Y}\}] = \frac{7}{18} + \frac{7}{9} = \frac{21}{18} = \frac{7}{6} = \frac{7}{12} + \frac{7}{12} = E[\mathbf{X}] + E[\mathbf{Y}]$ as it should be!

(h) $b = \rho\cdot\sqrt{\mathrm{var}(\mathbf{Y})/\mathrm{var}(\mathbf{X})} = \rho = \frac{-4}{11}$. $a = E[\mathbf{Y}] - bE[\mathbf{X}] = \frac{7}{12}\left[1 + \frac{4}{11}\right] = \frac{35}{44}$. Hence,

$\tilde{\mathbf{Y}} = \frac{35}{44} - \frac{4\mathbf{X}}{11} = \frac{35 - 16\mathbf{X}}{44}$ which is a straight line from (0,35/44) to (1,19/44). The graphs of the best estimate (a step function) and the best linear (straight line) estimate are as shown. Notice that the straight line estimate passes through the mean point (7/12,7/12) = ($E[\mathbf{X}]$,$E[\mathbf{Y}]$) while the step function estimate does not. This is a characteristic of best linear estimators: if $\mathbf{X}$ equals its mean, the best linear estimate of $\mathbf{Y}$ is the mean of $\mathbf{Y}$. Not always so for nonlinear optimum estimates. Notice also that the straight line estimate is a better approximation to the step function estimate in the interval (1/2, 1) because that is where most of the probability mass is!



The two estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ are the same at $\mathbf{X} = 1/8$ and $\mathbf{X} = 13/16$.

(i) Since $\hat{\mathbf{Y}}$ is a function of $\mathbf{X}$ (not $\mathbf{Y}$!) having value 3/4 if $\mathbf{X} < 1/2$ and value 1/2 if $\mathbf{X} > 1/2$, LOTUS allows us to write $E[(\mathbf{Y} - \hat{\mathbf{Y}})] = E[\mathbf{Y}] - E[\hat{\mathbf{Y}}] = \frac{7}{12} - \int_0^{1/2} (3/4)\,(2/3)\,du - \int_{1/2}^{1} (1/2)\,(4/3)\,du = \frac{7}{12} - \frac{1}{4} - \frac{1}{3} = 0$ so that the average error of the best mean-square estimate is 0. That $E[\mathbf{Y}] = E[\hat{\mathbf{Y}}]$ should be expected from Theorem 4.1 on p. 338 of the book. Similarly,

$$E[(\mathbf{Y} - \tilde{\mathbf{Y}})] = E[\mathbf{Y}] - E[\tilde{\mathbf{Y}}] = E[\mathbf{Y}] - E\left[\frac{35 - 16\mathbf{X}}{44}\right] = E[\mathbf{Y}] - \frac{35}{44} - \frac{16}{44} \times E[\mathbf{X}] = \frac{7}{12} - \frac{35}{44} - \frac{16}{44} \times \frac{7}{12} = 0.$$

On the other hand, $E[(\mathbf{Y} - \hat{\mathbf{Y}})^2] = \int_{v=1/2}^{1}\int_{u=0}^{1/2} (v-3/4)^2\cdot 4/3\,du\,dv + \int_{v=0}^{1}\int_{u=1/2}^{1} (v-1/2)^2\cdot 4/3\,du\,dv$

$= (2/3)\int_{v=1/2}^{1} (v-3/4)^2\,du\,dv + (2/3)\int_{v=0}^{1} (v-1/2)^2\,du\,dv = \frac{1}{16} = (0.25)^2$ while

$$E[(\mathbf{Y}-\tilde{\mathbf{Y}})^2] = E[(\mathbf{Y}-a-b\mathbf{X})^2] = \mathrm{var}(\mathbf{Y})\cdot[1-\rho^2] = \left(\frac{11}{144}\right)\left[1 - \left(\frac{-4}{11}\right)^2\right] = \frac{35}{528} \approx (0.257\ldots)^2 > \frac{1}{16} = (0.25)^2.$$

Ignoring $\mathbf{X}$ and estimating $\mathbf{Y}$ as $E[\mathbf{Y}] = \frac{7}{12}$ leads to mean-square error $\mathrm{var}(\mathbf{Y}) = \frac{11}{144} \approx (0.276\ldots)^2$

(j) Thank goodness!