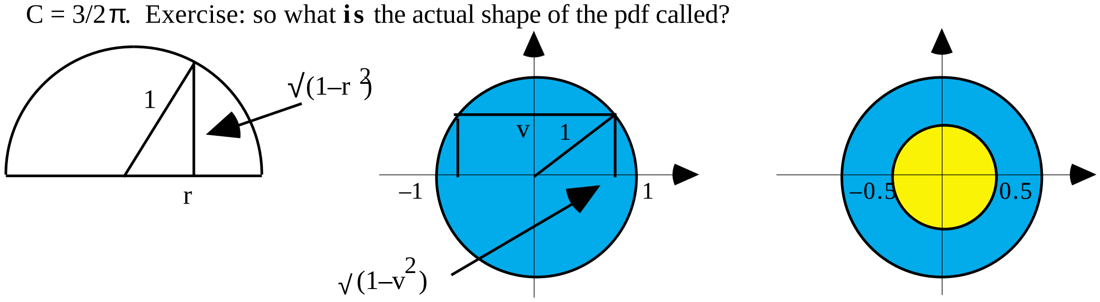
1.(a) $\sqrt{1-u^2-v^2}$ has value $\sqrt{1-r^2}$ at all points (u,v) on a circle of radius r < 1. Thus, as can be deduced from the left-hand figure below, the surface $\sqrt{1-u^2-v^2}$ is a hemisphere of radius 1, and thus has volume $2\pi/3$. Consequently, the pdf surface encloses a volume $2\pi C/3 = 1$ between itself and the u-v plane, giving that $C = 3/2\pi$. Exercise: so what **is** the actual shape of the pdf called?



For those who cannot think geometrically but prefer to act integrally, a direct evaluation of the volume begins by setting up the integral via the middle figure above. For fixed v, –1 < v < 1, u varies between $-\sqrt{1-v^2}$ and $+\sqrt{1-v^2}$. The double integral is converted to polar coordinates and evaluated as shown below.

$$\int_{v=-1}^{1}\int_{u=-\sqrt{1-v^2}}^{+\sqrt{1-v^2}} C\sqrt{1-u^2-v^2}\,du\,dv = \int_{r=0}^{1}\int_{\theta=0}^{2\pi} C\sqrt{1-r^2}\,r\,d\theta\,dr = 2\pi C\int_{r=0}^{1}\sqrt{1-r^2}\,r\,dr = -\frac{2\pi C}{3}(1-r^2)^{3/2}\Big|_{0}^{1} = \frac{2\pi C}{3}$$

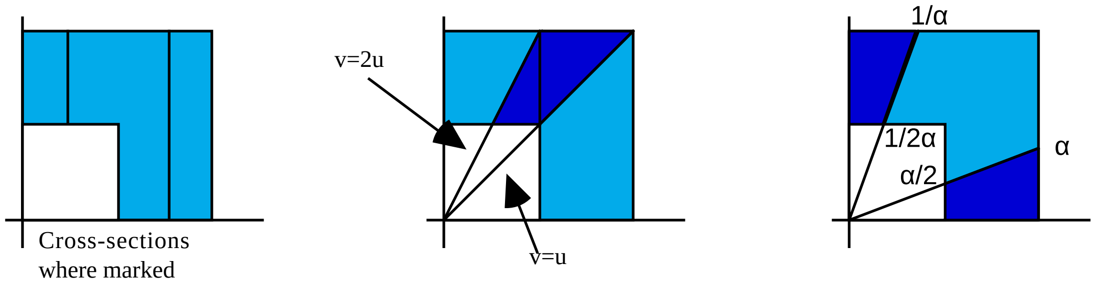
giving $C = 3/2\pi$ upon equating the volume to 1.

(b) $$P\{\mathbf{X}^2 + \mathbf{Y}^2 < 0.25\} = \int_{r=0}^{0.5}\int_{\theta=0}^{2\pi}(3/2\pi)\sqrt{1-r^2}\,r\,d\theta\,dr = 3\int_{r=0}^{0.5}\sqrt{1-r^2}\,r\,dr = -(1-r^2)^{3/2}\Big|_{0}^{0.5} = 1-\frac{3\sqrt{3}}{8}.$$

2.(a) The marginal pdf of **X** is the area of the cross-section of the joint pdf at the point u. There are two cases to be considered (0 < u < 1/2) and (1/2 < u < 1) as marked in the left-hand figure below. By inspection, we get that the area is (4/3)×(1/2) = 2/3 for 0 < u < 1/2, and (4/3)×1 = 4/3 for 1/2 < u < 1, and 0 for all other u.

Thus, $f_{\mathbf{X}}(u) = \begin{cases} 2/3, & 0 < u < 1/2,\\ 4/3, & 1/2 < u < 1,\\ 0, & \text{elsewhere.}\end{cases}$ It is easily checked that the area under the pdf is 1.

(b) From the symmetry of the joint pdf, it is obvious that $f_{\mathbf{Y}}(v) = \begin{cases} 2/3, & 0 < v < 1/2,\\ 4/3, & 1/2 < v < 1,\\ 0, & \text{elsewhere.}\end{cases}$



(c) The probability desired is the volume in the darkly shaded region shown in the middle figure above. We can easily get the area of this region as the sum of two triangles (1/2)×(1/2)×(1/4) + (1/2)×(1/2)×(1/2) = 3/16 and the probability is (4/3)×(3/16) = 1/4. Anti-segregationists (i.e. those who believe in integration) can get the same answer by writing

$$P\{\mathbf{X} < \mathbf{Y} < 2\mathbf{X}\} = \int_{v=1/2}^{1}\int_{u=v/2}^{v} 4/3\,du\,dv = \int_{v=1/2}^{1} 2v/3\,dv = v^2/3\Big|_{1/2}^{1} = \frac{1}{4}.$$

(d) The conditional pdf of **X** given **Y** = α is the cross-section $f_{\mathbf{X},\mathbf{Y}}(u,\alpha)$ of the joint pdf, normalized to have unit area. In this instance, it is obvious that the cross-section is a rectangle, and hence given **Y** = α, the conditional pdf of **X** is uniform on (1/2, 1) if 0 < α < 1/2, and uniform on (0, 1) if 1/2 < α < 1.

Thus, if $0 < \alpha < 1/2$, then

$$f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = \begin{cases} 2, & 1/2 < u < 1, \\ 0 & \text{otherwise,} \end{cases}$$

while if $1/2 < \alpha < 1$, then

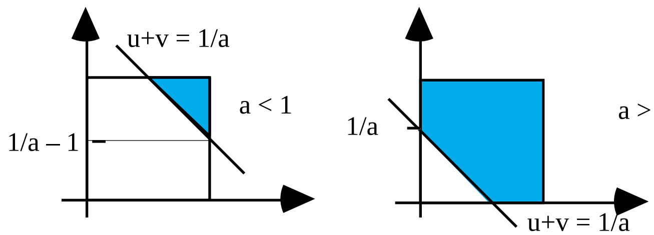
$$f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = \begin{cases} 1, & 0 < u < 1, \\ 0 & \text{otherwise,} \end{cases}$$

The theorem of total probability gives

$f_{\mathbf{X}}(u) = \int_0^1 f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) f_{\mathbf{Y}}(\alpha)d\alpha = \int_0^{1/2} f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)(2/3)d\alpha + \int_{1/2}^1 f_{\mathbf{X}|\mathbf{Y}}(u|\alpha)(4/3)d\alpha$ on substituting the numerical value of $f_{\mathbf{Y}}(\alpha)d\alpha$ from part (b).

Now, for *fixed* u, $0 < u < 1/2$, $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = 0$ if $0 < \alpha < 1/2$, and $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = 1$ if $1/2 < \alpha < 1$. Hence, the two integrals are 0 and (1/2)•(4/3), giving that $f_{\mathbf{X}}(u) = 2/3$ for all numbers u, $0 < u < 1/2$. On the other hand, for *fixed* u, $1/2 < u < 1$, $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = 2$ if $0 < \alpha < 1/2$, and $f_{\mathbf{X}|\mathbf{Y}}(u|\alpha) = 1$ if $1/2 < \alpha < 1$. Hence, the two integrals are (1/2)•2•(2/3) and (1/2)•(4/3), giving that $f_{\mathbf{X}}(u) = 4/3$ for all numbers u, $1/2 < u < 1$. This is the same answer as in part (b).

3. $\mathbf{I} = 1/(\mathbf{R}_1+\mathbf{R}_2)$ takes on values in the range $(1/2, \infty)$. Now, $F_{\mathbf{I}}(a) = P\{\mathbf{I} \le a\} = P\{1/(\mathbf{R}_1+\mathbf{R}_2) \le a\}$ $= P\{\mathbf{R}_1+\mathbf{R}_2 \ge a^{-1}\}$. Now, $1 < a < \infty$ implies that $a^{-1} < 1$ while $1/2 < a < 1$ implies that $1 < a^{-1} < 2$. Thus, we have the two cases to consider as illustrated in the figure below.



It follows readily from the diagrams that

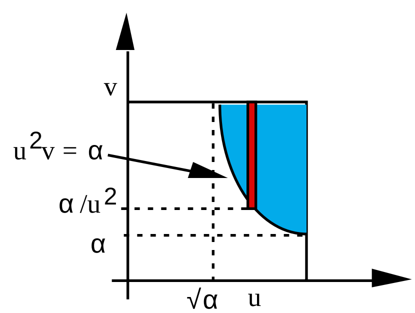
$$F_{\mathbf{I}}(a) = \begin{cases} 0, & a < 1/2, \\ (2 - a^{-1})^2/2, & 1/2 \le a \le 1, \\ 1 - a^{-2}/2, & 1 < a < \infty, \end{cases} \qquad f_{\mathbf{I}}(a) = \begin{cases} 2a^{-2} - a^{-3}, & 1/2 \le a \le 1, \\ a^{-3}, & 1 < a < \infty, \end{cases}$$

4. $\mathbf{Z} = \mathbf{X}^2\mathbf{Y}$. Then, $0 < \mathbf{Z} < 1$, and for $0 < \alpha < 1$,

$P\{\mathbf{Z} > \alpha\} = P\{\mathbf{X}^2\mathbf{Y} > \alpha\}$

$$= \int_{u=\sqrt{\alpha}}^{1} \int_{v=\alpha/u^2}^{1} 2u\,dv\,du = \int_{u=\sqrt{\alpha}}^{1} 2u(1 - \alpha/u^2)\,du$$

$$= u^2 - 2\alpha \cdot \ln u \Big|_{\sqrt{\alpha}}^{1} = (1 - \alpha) - (2\alpha \cdot \ln 1 - 2\alpha \cdot \ln\sqrt{\alpha}).$$



Hence, $1 - F_{\mathbf{Z}}(\alpha) = 1 - \alpha + \alpha \cdot \ln \alpha$ for $0 < \alpha < 1$, and the pdf $f_{\mathbf{Z}}(\alpha)$ is given by $-\ln \alpha$, $0 < \alpha < 1$, and 0 elsewhere. Quick check: Since $0 < \alpha < 1$, $\ln \alpha$ is negative, and hence $f_{\mathbf{Z}}(\alpha) > 0$ for $0 < \alpha < 1$. Furthermore, $F_{\mathbf{Z}}(\alpha) = \alpha - \alpha \cdot \ln \alpha$ approaches 1 as α approaches 1. In short, we have obtained a valid pdf.

5.(a) $f_{\mathbf{X}^2}(v) = \frac{1}{2\sqrt{v}} \cdot [f_{\mathbf{X}}(\sqrt{v}) + f_{\mathbf{X}}(-\sqrt{v})] = \frac{1}{2\sqrt{v}} \cdot \frac{1}{\sigma\sqrt{2\pi}} \cdot [\exp(-v/2\sigma^2) + \exp(-v/2\sigma^2)] = \frac{1}{\sigma\sqrt{2\pi v}} \cdot \exp(-v/2\sigma^2)$

$= \lambda \cdot \exp(-\lambda v) \cdot (\lambda v)^{1/2-1}/\Gamma(1/2)$ where $\lambda = 1/2\sigma^2$ and $\Gamma(1/2) = \sqrt{\pi}$ is a gamma pdf with parameters $(1/2, \lambda)$.

(b) The sum of independent gamma random variables with parameters (t_i, λ) is a gamma random variable with parameter $(\Sigma t_i, \lambda)$. Since $\mathbf{X}^2$, $\mathbf{Y}^2$, and $\mathbf{Z}^2$ are independent gamma random variables with parameter $(1/2, 1/2\sigma^2)$, $\mathbf{W} = \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$ is a gamma random variable with parameter $(3/2, 1/2\sigma^2)$. Its pdf is

$$f_{\mathbf{W}}(\alpha) = \begin{cases} \dfrac{\lambda(\lambda\alpha)^{t-1}\exp(-\lambda\alpha)}{\Gamma(t)}, & \alpha \geq 0, \\ 0, & \alpha < 0. \end{cases} = \begin{cases} \dfrac{1}{\sigma^3}\sqrt{\dfrac{\alpha}{2\pi}}\exp(-\alpha/2\sigma^2), & \alpha \geq 0, \\ 0, & \alpha < 0. \end{cases}$$

If $\sigma = 2$, $f_{\mathbf{W}}(5) = (1/8)\sqrt{5/2\pi}\exp(-5/8)$.

(c) $E[\mathbf{W}] = E[\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2] = E[\mathbf{X}^2] + E[\mathbf{Y}^2] + E[\mathbf{Z}^2] = 3\sigma^2$ because $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ are $N(0,\sigma^2)$ random variables and $E[\mathbf{X}^2] = \mathrm{var}(\mathbf{X}) + \mu^2 = \mathrm{var}(\mathbf{X}) = \sigma^2$ etc. We can check this by noting that a gamma random variable with parameter (t, λ) has mean t/λ. Shameless integrators can write

$$E[\mathbf{W}] = \int_0^\infty \frac{\alpha}{\sigma^3}\sqrt{\frac{\alpha}{2\pi}}\exp(-\alpha/2\sigma^2)d\alpha = \int_0^\infty \frac{4\sigma^2}{\sqrt{\pi}} x^{3/2}\exp(-x)dx = \frac{4\sigma^2}{\sqrt{\pi}}\Gamma\left(\frac{5}{2}\right) = \frac{4\sigma^2}{\sqrt{\pi}}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = 3\sigma^2$$

on substituting $2\sigma^2 x$ for α and remembering that $\Gamma(s) = (s-1)\Gamma(s-1)$ and that $\Gamma(1/2) = \sqrt{\pi}$.

(d) $E[\mathbf{H}] = E[(1/2)m\mathbf{W}] = (1/2)mE[\mathbf{W}] = (3/2)m\sigma^2 = (3/2)kT$, i.e $\sigma^2 = kT/m$. The pdf of $\mathbf{W}$ can thus be expressed as

$$f_{\mathbf{W}}(\alpha) = \begin{cases} \left(\dfrac{kT}{m}\right)^{-3/2}\sqrt{\dfrac{\alpha}{2\pi}}\exp(-m\alpha/2kT), & \alpha \geq 0, \\ 0, & \alpha < 0, \end{cases}$$

while the pdf of $\mathbf{H} = (m/2)\mathbf{W}$ is

$$f_{\mathbf{H}}(\beta) = \frac{2}{m}f_{\mathbf{W}}(2\beta/m) = \begin{cases} 2\sqrt{\dfrac{\beta}{\pi}}(kT)^{-3/2}\exp(-\beta/kT), & \beta \geq 0, \\ 0, & \beta < 0, \end{cases}$$

which is the Maxwell-Boltzmann pdf.

(e) $F_{\mathbf{V}}(\gamma) = P\{\mathbf{V} \leq \gamma\} = P\{\sqrt{\mathbf{W}} \leq \gamma\} = P\{\mathbf{W} \leq \gamma^2\} = F_{\mathbf{W}}(\gamma^2)$. Hence, $f_{\mathbf{V}}(\gamma) = \frac{d}{d\gamma}F_{\mathbf{V}}(\gamma) = \frac{d}{d\gamma}F_{\mathbf{W}}(\gamma^2)$ $= 2\gamma f_{\mathbf{W}}(\gamma^2)$ which gives the expression in the problem statement.

(f)

$$E[\mathbf{V}] = \int_0^\infty \gamma \cdot \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}\gamma^2\exp\left(-\frac{m\gamma^2}{2kT}\right)d\gamma = \int_0^\infty 2\sqrt{\frac{2kT}{m\pi}}x\cdot\exp(-x)dx = 2\sqrt{\frac{2kT}{m\pi}}$$

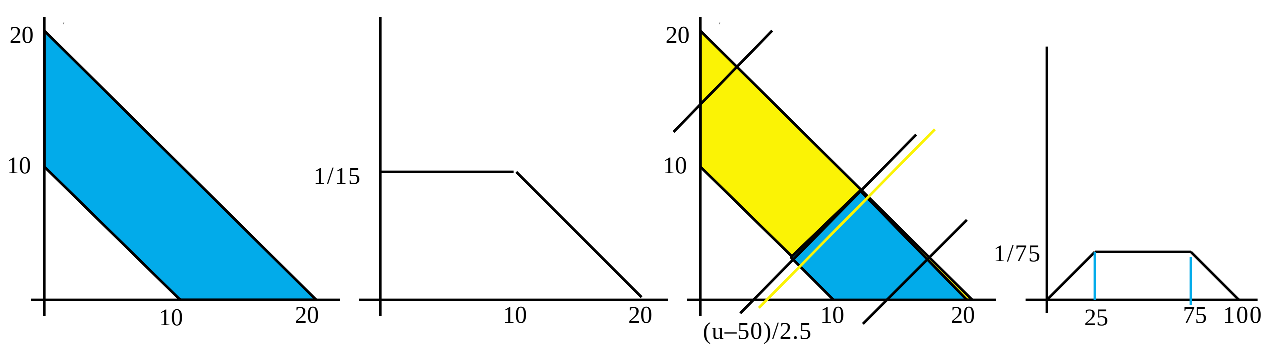
(Substitute $m\gamma^2/2kT = x$)

As a cross-check,

$$E[\mathbf{V}] = E[\sqrt{\mathbf{W}}] = \int_0^\infty \sqrt{\alpha}\cdot\frac{1}{\sigma^3}\sqrt{\frac{\alpha}{2\pi}}\exp(-\alpha/2\sigma^2)d\alpha = \int_0^\infty \frac{4\sigma}{\sqrt{2\pi}}x\cdot\exp(-x)dx = 2\sqrt{\frac{2kT}{m\pi}}$$

on substituting $\alpha/2\sigma^2 = x$.

6.(a) The joint pdf has constant value K on the shaded region shown in the left-hand figure below. The area can be computed as the difference in area of two triangles as (1/2)•20•20 – (1/2)•10•10 = 150, giving K = 1/150.



(b) $f_{\mathbf{R}}(x)$, the marginal pdf of $\mathbf{R}$ is the cross-sectional area of the joint pdf at x. Now, the cross-section is always a rectangle of height 1/150, and its base is 10 if $0 < x < 10$, while the base is $20-x$ if $10 \leq x < 20$.

Hence, we get that $f_{\mathbf{R}}(x) = \begin{cases} 1/15, & 0 < x < 10, \\ (20-x)/150, & 10 < x < 20, \\ 0, & \text{elsewhere.} \end{cases}$ which is shown in the middle figure above.

Note that P{student spends less than 10 hours studying for the ECE 313 Final} = 2/3.

(c) **T** takes on values from 0 (when **S** = 0 and **R** = 20, that is, the student is sleepless and hopelessly confused) to 100 (when **S** = 20 and **R** = 0, that is, the student is well-rested and relies entirely on work done during the semester to carry him/her through the Final Exam). For any u, $0 < u < 100$,
$P\{\mathbf{T} \le u\} = P\{\mathbf{S} - \mathbf{R} \le (u-50)/2.5\} = (1/150)\cdot$shaded trapezoidal area in third figure above, but note that there are special cases to be considered when $u < 25$ and $u > 75$. We can now proceed to find $P\{\mathbf{T} \le u\} = F_{\mathbf{T}}(u)$ and differentiate to find the pdf, but an easier argument is as follows.
For $25 \le u < 75$, and *small* δu, $P\{u \le \mathbf{T} \le u+\delta u\} \approx f_{\mathbf{T}}(u)\cdot\delta u$
= volume between the two lines close together in the diagram = $(1/150)\cdot(10/\sqrt{2})\cdot(\delta u/\sqrt{2})$, since that thin sliver has height (1/150), and the sides of the base are $10/\sqrt{2}$ and $(\delta u/2.5)/\sqrt{2}$ respectively. We conclude that $f_{\mathbf{T}}(u) = 1/75$ for $25 < u < 75$. A similar argument shows that the pdf of **T** increases linearly from 0 at u = 0 to 1/75 at u = 25, and decreases from 1/75 at u = 75 to 0 at u = 100. In short, it looks as shown above in the rightmost figure..