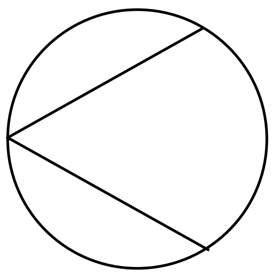
Assigned: Wednesday, October 31, 2001
Due: Wednesday, November 7, 2001
Reading: Ross, Chapter 5 and Chapter 6
Noncredit Exercises: Ross, Chapter 5: Problems 15-38; Chapter 6: Problems 1, 8-15, 20-23
Problems:

1. [Read Example 3d on pp. 198-199 first.] Let the (straight) line segment ACB be a diameter of a circle of unit radius and center C. Consider an arc AD of the circle where the length **X** of the *arc* (measured clockwise around the circle) is a random variable uniformly distributed on $[0,2\pi)$. Now consider the "random chord" AD.

(a) Find the probability that the length **L** of the random chord is greater than the side of the equilateral triangle inscribed in the circle.

(b) Express **L** as a function of the random variable **X**, and find the probability density function for **L**.



1.(a) As is obvious from the figure, the chord is longer than the side of the inscribed equilateral triangle if $2\pi/3 < \mathbf{X} < 4\pi/3$. Hence, the desired probability is $(4\pi/3-2\pi/3)/2\pi = 1/3$ as in the second model in Ross. What is the geometrical relation between the two models?

(b) Since the circle has radius 1, an arc of length θ subtends an angle θ at C. Also, the length of the chord joining the endpoints of the arc is $2\sin(\theta/2)$. Hence, $\mathbf{L} = 2\sin(\mathbf{X}/2)$. Note that as **X** increases from 0 to 2π, the chord length increases from 0 to 2 (at $\mathbf{X} = \pi$), and then decreases to 0 (at $\mathbf{X} = 2\pi$). For any x, $0 \le x \le 2$, $F_{\mathbf{L}}(x) = P\{\mathbf{L} \le x\} = P\{2\sin(\mathbf{X}/2) \le x\} = 2P\{0 \le \mathbf{X} \le 2\arcsin(x/2)\}$ (Why twice?) $= 2(2\arcsin(x/2)/2\pi) = (2/\pi)\arcsin(x/2)$.

Hence, $f_{\mathbf{L}}(x) = \frac{d}{dx}F_{\mathbf{L}}(x) = \begin{cases} \dfrac{1}{\pi\sqrt{1-(x/2)^2}} & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$

2. The random variable **X** has probability density function $f_{\mathbf{X}}(u) = \begin{cases} 2(1-u), & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

Let $\mathbf{Y} = (1-\mathbf{X})^2$.

(a) What is the CDF $F_{\mathbf{Y}}(v)$ of the random variable **Y**? Be sure to specify the value of $F_{\mathbf{Y}}(v)$ for all v, $-\infty < v < \infty$.

(b) Show that the $F_{\mathbf{Y}}(v)$ that you found in part (b) is a nondecreasing continuous function.

2.(a) Let $0 \le v \le 1$. Then, $F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \le v\} = P\{(1-\mathbf{X})^2 \le v\} = P\{-\sqrt{v} \le 1-\mathbf{X} \le \sqrt{v}\} = P\{\mathbf{X} \ge 1-\sqrt{v}\}$ $= 1 - F_{\mathbf{X}}(1-\sqrt{v}) = (1-(1-\sqrt{v}))^2 = v$ where we used the result that $F_{\mathbf{X}}(u) = 1-(1-u)^2$.

Hence, $F_{\mathbf{Y}}(v) = \begin{cases} 0, & u < 0, \\ v, & 0 \le v \le 1, \\ 1, & v > 1. \end{cases}$

(b) A sketch of the function $F_{\mathbf{Y}}(v)$ reveals that it is a nondecreasing continuous function. It is not differentiable at $\alpha = 0$ or at $\beta = 1$.

3. The radius of a sphere is a random variable **R** with pdf $f_{\mathbf{R}}(\rho) = \begin{cases} 3\rho^2, & 0 < \rho < 1, \\ 0 & \text{elsewhere.} \end{cases}$

(a) Use LOTUS to find the average radius, average volume and average surface area of the sphere. Does a sphere of average radius have average volume? Does a sphere of average radius have average surface area?

(b) Find the CDF $F_{\mathbf{V}}(\alpha)$ and pdf $f_{\mathbf{V}}(\alpha)$ of **V**, the volume of the sphere.

(c) Find E[**V**] directly from this pdf. Do you get the same answer as in part (a)? Why not?

(d) If the sphere is made of metal and carries an electrical charge of Q coulombs, what is the CDF $F_{\mathbf{S}}(x)$ and the pdf $f_{\mathbf{S}}(x)$ of the surface charge density **S** on the sphere ?

3.(a) $E[\mathbf{R}] = \int_0^1 3\rho^3 d\rho = \frac{3}{4}$; $E[\mathbf{V}] = E[4\pi\mathbf{R}^3/3] = \int_0^1 4\pi\rho^5 d\rho = \frac{2\pi}{3}$; $E[\mathbf{A}] = E[4\pi\mathbf{R}^2] = \int_0^1 12\pi\rho^4 d\rho = \frac{12\pi}{5}$.

The average volume $E[\mathbf{V}] = E[4\pi\mathbf{R}^3/3]$ corresponds to a sphere of radius $(1/2)^{1/3}$ and the average surface area $E[\mathbf{A}] = E[4\pi\mathbf{R}^2]$ to a sphere of radius $(3/5)^{1/2}$. Note that $E[\mathbf{V}] = E[4\pi\mathbf{R}^3/3] \neq 4\pi(E[\mathbf{R}])^3/3$, etc. This illustrates the general result that $E[g(\mathbf{X})]$ hardly ever equals $g(E[\mathbf{X}])$. You will save yourself a lot of grief if you keep this in mind: that $E[g(\mathbf{X})] = g(E[\mathbf{X}])$ is a common misconception among the instochaste.

Exercise: Find a function $g(\bullet)$ for which $E[g(\mathbf{X})]$ *does* equal $g(E[\mathbf{X}])$.

(b) The volume V has values in the range $(0, 4\pi/3)$. For any u, $0 < u < 4\pi/3$, $F_{\mathbf{V}}(u) = P\{\mathbf{V} \le u\}$
$= P\{4\pi\mathbf{R}^3/3 \le u\} = P\{\mathbf{R} \le \sqrt[3]{3u/4\pi}\} = F_{\mathbf{R}}(\sqrt[3]{3u/4\pi}) = 3u/4\pi$ since $F_{\mathbf{R}}(\rho) = \rho^3$ for $0 < \rho < 1$. Hence, $f_{\mathbf{V}}(u)$ is uniform on $(0, 4\pi/3)$.

(c) Obviously, $E[\mathbf{V}]$ = midpoint of uniform pdf = $2\pi/3$ as in part (a)

(c) The electrical charge is uniformly distributed on the surface of the sphere. The surface charge density is $\mathbf{S} = Q/4\pi\mathbf{R}^2 > Q/4\pi$.
For $x > Q/4\pi$, $F_{\mathbf{S}}(x) = P\{\mathbf{S} \le x\} = P\{Q/4\pi\mathbf{R}^2 \le x\} = P\{1 > \mathbf{R} \ge \sqrt{Q/4\pi x}\} = 1 - (Q/4\pi x)^{1.5}$.
Hence, $f_{\mathbf{S}}(x) = (3/2x)(Q/4\pi x)^{1.5}$ for $x > Q/4\pi$, and 0 otherwise.

4. ["Give me an A! Give me a D! Give me a converter! What have you got? An A/D converter! Go Team!"]
A signal **X** is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value **Y** (where $\mathbf{Y} = \alpha$ if $\mathbf{X} > 0$ and $\mathbf{Y} = -\alpha$ if $\mathbf{X} \le 0$) is used. Note that **Y** is a *discrete* random variable.

(a) What is the pmf of **Y**?

(b) Suppose that $\alpha = 1$. If the signal **X** happens to have value 1.29, what is the error made in representing **X** by **Y**? What is the squared-error? Repeat for the case when **X** happens to have value $\pi/4$ and when **X** happens to have value $-\pi/4$.

(c) We wish to design the quantizer so as to minimize the squared-error. However, since **X** (and **Y**) are random, we can only minimize the squared-error in the probabilistic (that is, average) sense. Now, part (b) shows that the squared-error depends on the value of **X**, and can be expressed as
$$\mathbf{Z} = (\mathbf{X} - \mathbf{Y})^2 = g(\mathbf{X}) = \begin{cases} (\mathbf{X} - \alpha)^2 & \text{if } \mathbf{X} > 0 \\ (\mathbf{X} + \alpha)^2 & \text{if } \mathbf{X} \le 0. \end{cases}$$
So we want to choose α so that $E[\mathbf{Z}]$ is as small as possible. Use LOTUS to e-zily find $E[\mathbf{Z}]$ as a function of α, and then find the value of α that minimizes $E[\mathbf{Z}]$.

(d) We now get more ambitious and use a 3-bit A/D converter which first quantizes **X** to the nearest integer **W** in the range –3 to +3. Thus, $\mathbf{W} = 3$ if $\mathbf{X} \ge 2.5$, $\mathbf{W} = 2$ if $1.5 \le \mathbf{X} < 2.5$, etc. Note that **W** is a discrete random variable. Find the pmf of **W**.

(e) The output of the A/D converter is a 3-bit 2's complement representation of **W**. Suppose that the output is $(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_0)$. What is the pmf of $\mathbf{Z}_2$? of $\mathbf{Z}_1$? of $\mathbf{Z}_0$?

(f) **Noncredit exercise (but a real-life engineering problem!):** Suppose that **W** takes on values $-3\alpha, -2\alpha, -\alpha, 0, +\alpha, +2\alpha, +3\alpha$ and quantization is as before: **X** is mapped to the nearest **W** value. What value of α minimizes $E[(\mathbf{X} - \mathbf{W})^2]$?

4.(a) Obviously $P\{\mathbf{Y} = \alpha\} = P\{\mathbf{Y} = -\alpha\} = 1/2$.

(b) $(1.29-1) = 0.29$. $(1.29-1)^2 = 0.0841$. $(\pi/4-1) = -0.214\ldots$, $(\pi/4-1)^2 = 0.046\ldots$.
$(-\pi/4-(-1)) = -0.214\ldots$, $(-\pi/4-(-1))^2 = 0.046\ldots$. Note that the error for +**X** is the same as that for –**X**.

(c) $E[\mathbf{Z}] = \int_0^\infty (u-\alpha)^2 f(u)du + \int_{-\infty}^0 (u+\alpha)^2 f(u)du = \int_{-\infty}^\infty (u^2 + \alpha^2) f(u)du - 4\alpha\int_0^\infty u f(u)du = 1+\alpha^2-4\alpha/\sqrt{2\pi}$ on expanding out the quadratics, changing variables, and using the fact that $E[\mathbf{X}^2] = \sigma^2 + \mu^2 = 1$. Note that $uf(u)$ is a perfect integral. It is easy to show that $E[\mathbf{Z}]$ has minimum value $1-2/\pi$ at $\alpha = \sqrt{2/\pi}$

(d) From tables of $\Phi(\bullet)$, we get $P\{\mathbf{W} = -3\} = P\{\mathbf{W} = +3\} = \Phi(-2.5) = 0.0062$,
$P\{\mathbf{W} = 0\} = \Phi(0.5) - \Phi(-0.5) = 0.3830$, $P\{\mathbf{W} = -1\} = P\{\mathbf{W} = +1\} = \Phi(1.5) - \Phi(0.5) = 0.2417$, and
$P\{\mathbf{W} = -2\} = P\{\mathbf{W} = +2\} = \Phi(2.5) - \Phi(1.5) = 0.0606$.

(e) $P\{\mathbf{Z}_2 = 1\} = P\{\mathbf{W} < 0\} = 0.3085$.
$P\{\mathbf{Z}_1 = 1\} = P\{\mathbf{W} = 2\} + P\{\mathbf{W} = 3\} + P\{\mathbf{W} = -1\} + P\{\mathbf{W} = -2\} = 0.3691$
$P\{\mathbf{Z}_0 = 1\} = P\{\mathbf{W} = 2\} + P\{\mathbf{W} = 0\} + P\{\mathbf{W} = -2\} = 0.5042 \neq P\{\mathbf{Z}_0 = 0\}$

5. The lifetime of a system with hazard rate $\lambda(t) = bt$ is a Rayleigh random variable $\mathbf{X}$ with pdf $f(u) = (bu)\bullet\exp(-bu^2/2)$ for $u > 0$ (Ross, p. 216). The system fails at time t, i.e. $\mathbf{X} = t$ is observed to have occurred on this trial. What is the maximum-likelihood estimate of the parameter b that occurs in the pdf and hazard rate? Remember that the maximum-likelihood estimate $\hat{b}$ of the parameter b maximizes the pdf at the observed value t. Thus, for given t, what value of b maximizes $(bt)\exp(-bt^2/2)$?

5. $\frac{d}{db}(bt)\exp(-bt^2/2) = t\bullet\exp(-bt^2/2) - bt\bullet\exp(-bt^2/2)\bullet t^2/2$ is zero for $b = \sqrt{2}/t$. Thus, if we observe that $\mathbf{X} = t$, the maximum-likelihood estimate of b is $\sqrt{2}/t$. Reality check: If the observed value t is large, we estimate the value of b to be quite small. This makes sense. If the system lasted for a long time, its hazard rate can be expected to be small, and the hazard rate is proportional to b.

6. If hypothesis H_0 is true, the pdf of $\mathbf{X}$ is $f_0(u) = (1/2)\exp(-|u+1|)$, $-\infty < u < \infty$, while if hypothesis H_1 is true, the pdf of $\mathbf{X}$ is $f_1(u) = (1/2)\exp(-|u-1|)$, $-\infty < u < \infty$.
Such pdfs are called LaPlacian or double exponential pdfs.

(a) Sketch the two pdfs.
Be careful: those absolute-value signs are trickier than they look!

(b) State the maximum-likelihood decision rule in terms of a threshold test on the observed value u of the random variable $\mathbf{X}$ instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_0(u)$ with 1.

(c) What are the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part (b)?

(d) Compute the values of the likelihood ratio for $u = -1.2, -1, -0.8, \ldots, 0.8, 1, 1.2$.

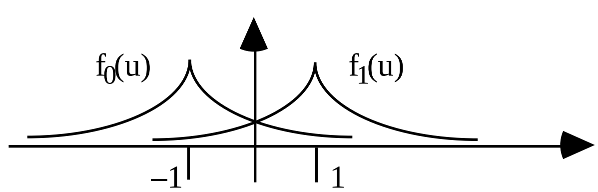
(e) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to (π_0/π_1). Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable $\mathbf{X}$.

(f) If $\pi_0 = 2\pi_1$, what is the average probability of error of the Bayesian decision rule?

(g) What is the average error probability of a decision rule that always decides H_0 is the true hypothesis, regardless of the value taken on by $\mathbf{X}$?

(h) Show that if $\pi_0 > e^2/(e^2+1)$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by $\mathbf{X}$.

6.(a)



(b) The maximum-likelihood decision chooses the hypothesis which has the larger pdf value at the observation. Here, by inspection of the answer to (a), we see that the decision is to choose H_1 if $\mathbf{X} > 0$ and H_0 if $\mathbf{X} < 0$.

(c) P_{FA}= P{false alarm} = P{H_1 is chosen when in fact H_0 is the true hypothesis} = P{$\mathbf{X} > 0$ when H_0 is true}

$$= \int_0^\infty f_0(u)du = \int_0^\infty (1/2)\bullet\exp(-|u+1|)du = \int_0^\infty (1/2)\bullet\exp(-u-1)du = (1/2)\exp(-1)\int_0^\infty \exp(-u)du = (1/2)\bullet\exp(-1).$$

Similarly, P_{MD} = P{missed detection} = $(1/2)\bullet\exp(-1)$ also.

(d) $\Lambda(u) = \exp(-|u-1|)/\exp(-|u+1|) = \begin{cases} \exp(2), & u > 1, \\ \exp(2u), & -1 \le u \le 1, \\ \exp(-2), & u < -1. \end{cases}$ I told you those absolute-value signs were tricky! $\Lambda(u) = \exp(\pm 2)$ if $u = \pm 1.2, \pm 1$; $\Lambda(u)$ increases from $\exp(-2)$ to $\exp(2)$ as u increases from –1 to 1.

(e) Assuming that $\exp(-2) < (\pi_0/\pi_1) < \exp(2)$, $\Lambda(u) > \pi_0/\pi_1$ for $u > (1/2)\ln(\pi_0/\pi_1) = \ln\left(\sqrt{\pi_0/\pi_1}\right)$. Hence, the Bayesian decision rule is to choose H_1 if $\mathbf{X} > \ln\left(\sqrt{\pi_0/\pi_1}\right)$ and H_0 if $\mathbf{X} < \ln\left(\sqrt{\pi_0/\pi_1}\right)$. On the other hand, if $(\pi_0/\pi_1) > \exp(2)$, then $\Lambda(u)$, which has maximum value $\exp(2)$, can never exceed (π_0/π_1) and the Bayesian decision is to always decide that H_0 is the true hypothesis. Similarly, if $(\pi_0/\pi_1) < \exp(2)$, then $\Lambda(u)$, which has minimum value $\exp(-2)$, can never be smaller than (π_0/π_1) and the Bayesian decision is to always decide that H_1 is the true hypothesis.

(f) If $\pi_0 = 2\pi_1 = 2/3$, the Bayesian decision chooses H_1 whenever $\mathbf{X} > \ln\left(\sqrt{\pi_0/\pi_1}\right) = \ln(\sqrt{2}) = \theta > 0$.

$$P_{FA} = \int_\theta^\infty (1/2)\bullet\exp(-|u+1|)du = \int_\theta^\infty (1/2)\bullet\exp(-u-1)du = (1/2)\exp(-1)\int_\theta^\infty \exp(-u)du = (1/2)\bullet\exp(-1-\theta) = 1/(2\sqrt{2}e).$$

$$\text{Similarly, } P_{MD} = \int_{-\infty}^\theta (1/2)\bullet\exp(-|u-1|)du = \int_{-\infty}^\theta (1/2)\bullet\exp(u-1)du = (1/2)\exp(-1)\int_{-\infty}^\theta \exp(u)du = (1/2)\bullet\exp(-1+\theta)$$

$= 1/(\sqrt{2}e) = 2P_{FA}$. Finally, the average error probability is $\pi_0 P_{FA} + \pi_1 P_{MD} = \sqrt{2}/3e$. More generally, the average error probability is $\sqrt{\pi_0\pi_1}\exp(-1)$ which has maximum value (1/2))exp(–1) if $\pi_0 = \pi_1 = 1/2$. Of course, all the above applies only if $\exp(-2) < (\pi_0/\pi_1) < \exp(2)$.

(g) The decision rule that always chooses H_0 makes an error precisely in those instances when H_1 is the true hypothesis. Hence its average error probability is just π_1, the probability that H_1 is the true hypothesis.

(h) If $\pi_0 > \exp(2)/[\exp(2)+1]$, then $\pi_1 = 1 - \pi_0 < 1/[\exp(2)+1]$, and thus $(\pi_0/\pi_1) > \exp(2)$. It follows that the likelihood ratio, which has maximum value exp(2) can never exceed π_0/π_1 and hence the Bayesian decision rule is to always decide that H_0 is the true hypothesis. The error probability is thus $\pi_1 < 1/[\exp(2)+1]$.

7. The random variable **X** models a physical parameter. If hypothesis H_0 is true, then, $f_0(u)$, the pdf of **X**, is Gaussian with mean 0 and variance a^2. On the other hand, if hypothesis H_1 is true, then $f_1(u)$, the pdf of **X**, is Gaussian with mean 0 and variance $b^2 > a^2$.

(a) Suppose that H_0 and H_1 have equal probability. Thus, for i = 0, 1, the pdf of **X** when hypothesis H_i is true can be thought of as the *conditional* pdf of **X** given that H_i occurred, i.e. $f_{\mathbf{X}|H_i}(u|H_i)$. Write an expression for the *unconditional* pdf of **X**. Is the unconditional pdf of **X** a Gaussian pdf?

(b) What is the likelihood ratio? Simplify your answer.

(c) What is the maximum-likelihood decision rule, and what are the false alarm probability and the missed detection probability of this rule?

7.(a) No, the unconditional pdf of **X** is given by $[(a\sqrt{2\pi})^{-1}\exp(-u^2/2a^2) + (b\sqrt{2\pi})^{-1}\exp(-u^2/2b^2)]/2$, which is not a Gaussian pdf.

(b) $$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{(a\sqrt{2\pi})\exp(-u^2/2b^2)}{(b\sqrt{2\pi})\exp(-u^2/2a^2)} = \frac{a}{b}\bullet\exp\frac{-u^2}{2}\left[\frac{1}{b^2} - \frac{1}{a^2}\right]$$

(c) Suppose that the observation **X** has value u. The maximum-likelihood decision rule says that H_1 is chosen as the true hypothesis if $\Lambda(u) > 1$ and H_0 is chosen if $\Lambda(u) < 1$. Thus, H_1 is chosen if

$$\ln(a/b) - (u^2/2)(b^{-2} - a^{-2}) > 0.$$

This is equivalent to the statement that the rule chooses H_1 whenever the observation **X** is such that

$$|\mathbf{X}| > ab\sqrt{\frac{\ln b^2 - \ln a^2}{b^2 - a^2}} = c.$$

Note that $f_0(0) = 1/(a\sqrt{2\pi}) > 1/(b\sqrt{2\pi}) = f_1(0)$ and the two pdf curves cross each other at $\pm c$.
P(false alarm) = $P\{|\mathbf{X}| > c \mid H_0 \text{ is true}\} = 2Q(c/a)$. P(missed detection) = $P\{|\mathbf{X}| < c \mid H_1 \text{ is true}\} = 1 - 2Q(c/b)$.