1.(a) The number of arrivals in the interval (0, 4] is the Poisson random variable **N**(0,4] with parameter 4λ. Hence, the mean number of arrivals is E[**N**(0,4]] = 4λ.

(b) $P[\{\mathbf{N}(0,3]=3\}\cap\{\mathbf{N}(2,6]=0\}] = P[\{\mathbf{N}(0,2]=3\}\cap\{\mathbf{N}(2,6]=0\}] = \left[\exp(-2\lambda)\frac{(2\lambda)^3}{3!}\right]\times\exp(-4\lambda)$

$=\exp(-6\lambda)\frac{(2\lambda)^3}{3!}$ since **N**(0, 2] and **N**(2, 6] are independent random variables ((0,2] and (2,6] are disjoint)

(c) The number of arrivals in (0, 6] is the Poisson random variable **N**(0,6] with parameter 6λ. The event {**N**(0,6] = 5} has probability $\exp(-6\lambda)\frac{(6\lambda)^5}{5!}$, and this is maximized if λ = 5/6 as shown on a noncredit exercise in Problem Set #5.

(d) λ = ln 2. $P\{\mathbf{N}(0,t]\geq 1\} = 1 - P\{\mathbf{N}(0,t]=0\} = 1-\exp(-\lambda t) = 1-2^{-t}$.

2.(a) They would collide all over again in slot #(n+1). This could continue for ever with the collision never getting resolved. Note that "Try ten times and quit if it still doesn't go through" merely wastes the next ten slots. Thus, some mechanism is needed to prevent repeated collisions, and randomization of the next time that a backlogged packet is transmitted is one method for doing this. Note: rolling a die will not work unless the chances of six or more collisions is very small.

(b) P{no backlogged transmitter sends a packet} = P{both dice result in numbers > 1} = $(5/6)^2$ = 25/36.
P{successful re-transmission} = P{only one die results in 1} = 2•(1/6)(5/6) = 10/36.
P{both transmissions collide again} = P{both dice result in 1} = $(1/6)^2$ = 1/36.

(c) **X** equals 2 if and only if the dice rolls resulted in (1,2) or (2,1) which has probability 2/36.
X equals 3 if the dice rolls resulted in (1,3), (2,3), (3,1), (3,2) **or** the first rolls resulted in (1,1) so that there was a collision in slot #(n+1) too, and the **next** rolls resulted in (1,2) or (2,1) giving a resolution in slot #(n+3). Hence P(**X** = 3} = 4/36 + (1/36)P{**X** = 2} = 4/36 + (1/36)(2/36) = $146/36^2$. **X** equals 4 if the first dice rolls resulted in (1,4), (2,4), (3,4), (4,1), (4,2), (4,3) **or** the first rolls resulted in (1,1) and then 3 more slots were needed to resolve the collisions **or** the first rolls resulted in (2,2) and then 2 more slots were needed to resolve the collisions. Thus, P{**X** = 4} = 6/36 + (1/36)•P{**X** = 3} + (1/36)•P{**X** = 2}.

(d) P{P{no backlogged transmitter sends a packet} = P{three dice result in numbers > 1} = $(5/6)^3$ = 125/216.
P{successful re-transmission} = P{only one die results in 1} = 3•(1/6)(5/6)(5/6) = 75/216.
P{all three transmissions collide again} = P{all three dice result in 1} = $(1/6)^3$ = 1/216.
P{exactly two transmissions collide again} = P{two dice result in 1} = 3•$(1/6)^2$(5/6) = 15/216.
Now P{**X** = 2} = 0 (why?) while P{**X** = 3} = P{three dice result in 1,2,3} = 6•$(1/6)^3$ = 1/36. (why 6• ??)
On the other hand, **X** equals 4 if the rolls resulted in (1,2,4), (1,3,4), (2,3,4) or any permutation thereof (18 choices total) **or** there was a (two- or three-packet) collision in slot #(n+1) and then three slots sufficed to resolve the collision **or** the first roll resulted in (1,2,2) or any permutation thereof, and then the two transmitters colliding in slot #(n+2) rolled (1,2) or (2,1) and thus resolved matters by slot #(n+4). This has probability (18/216) + (16/216)(1/36) + (3/216)(2/36). Exercise: where did 16/216 come from?

(e) It is reasonable to assume that the N transmitters act independently in deciding whether to send a packet or not. Thus, the number of packets transmitted in a slot can be modeled a binomial random variable with parameters (N, p) where N is large and p is very small, and this can be approximated as a Poisson random variable with parameter λ = Np. Since E[Poisson] = λ, λ is called the *offered traffic*: it is the *average number of packet transmissions attempted per slot.* Typically the system analyst counts the total number of transmission attempts in a large number of slots (10,000, say) and sets λ to be this total divided by 10,000 (instead of going all around the campus counting the active terminals and deciding on the value of p!)

(f) p = probability that a *successful* packet transmission occurs = P{one packet transmitted} = λ•exp(–λ). In each slot, either no transmission occurs, or one successful packet transmission occurs (with probability p) or multiple transmissions occur, all of which are unsuccessful. Thus, **Y**, the number of *successful* transmissions in a slot is a Bernoulli random variable with parameter p. Note that E[**Y**] = p = λ•exp(–λ). E[**Y**] is called the *throughput* of the system: it is the *average number of successful transmissions per slot.* What offered traffic rate maximizes the throughput? It is readily verified that the maximum throughput is exp(–1) ≈ 0.36… which occurs when λ = 1. For offered traffic rates less than 1, many slots will have no transmissions at all, thus reducing the throughput. For offered traffic rates more than 1, too many slots will have multiple transmissions, all of which will be unsuccessful, and this will reduce the throughput.

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* so called because it was invented at the University of Hawaii for connecting the computer terminals on their campus to a central mainframe. The actual slotted ALOHA collision resolution algorithm is too difficult to analyze in homework in this course. The method described above is simpler to analyze, but does not capture most of the nuances. A more detailed (and correct!) description of the actual system is often taught in the course ECE/CS 338 *Communication Networks for Computers* (which has ECE 313 as one of its prerequisites).

3.(b)(c) Eq. 26.2.17 is the most commonly used approximation to Q(x). It gives $Q(5) \approx 2.8710\times10^{-7}$. The error in the approximation is known to be smaller than 7.5×10^{-8} so that the actual value of Q(5) *could be* as small as $2.8710\times10^{-7} - 7.5\times10^{-8} = 2.1210\times10^{-7}$ or it *could be* as large as $2.8710\times10^{-7} + 7.5\times10^{-8} = 3.6210\times10^{-7}$ and thus the maximum relative error *could be* as large as $7.5\times10^{-8}/2.1210\times10^{-7} \approx 35.4\%$.

(d) $-\log_{10}Q(5) = 6.54265$, so $Q(5) = 2.8667.5\times10^{-7}$. Thus, even though the error specification implies that there *could be* a 35% error, Eq. 26.2.17 actually overestimates Q(5) by only 0.16%! In contrast, Eq. (4.4) is $2.8545\times10^{-7} < Q(5) < 2.973410^{-7}$. The upper bound overestimates Q(5) by 3.57% and the lower bound underestimates Q(5) by 0.42%. Thus, at x = 5, both the upper bound and the lower bound of Eq. (4.4) of Ross are poorer approximations to Q(5) than the value given by Eq. (26.2.17) of Abramowitz and Stegun.

(e) The maximum possible error is larger than the value being computed.

4.(a) $d/du(\exp-u^2/2)) = -u\exp(-u^2/2)$. Hence,

$$E[|\mathbf{X}|] = \int_{-\infty}^{\infty}|u|\bullet\varphi(u)du = 2\int_0^{\infty}u\bullet\varphi(u)du = 2\int_0^{\infty}u\bullet\frac{1}{\sqrt{2\pi}}\exp(-u^2/2)du = \sqrt{\frac{2}{\pi}}\bullet -\exp(-u^2/2)\Big|_0^{\infty} = \sqrt{\frac{2}{\pi}} < 1.$$

More generally, $E[|\mathbf{X}-\mu|] = \sigma\sqrt{\frac{2}{\pi}}$ if the pdf of **X** is $N(\mu,\sigma^2)$. In statistical applications, $E[|\mathbf{X}-\mu|]$ is sometimes called the absolute error.

(b) Since $\Phi(x) = 0.9999...$ etc for large x, the noninformative 9's waste paper, whereas Q(x) can be specified more easily as, for example, $Q(5) = 2.8665\times10^{-7}$.

(c) $$Q(x) = \int_x^{\infty}\varphi(u)\,du = \int_x^{\infty}u^{-1}u\varphi(u)\,du = x^{-1}\varphi(x) - \int_x^{\infty}u^{-2}\varphi(u)\,du = x^{-1}\varphi(x) - \int_x^{\infty}u^{-3}\;u\;\;\varphi(u)\,du$$

$$= x^{-1}\varphi(x) - x^{-3}\varphi(x) + \int_x^{\infty}3u^{-4}\varphi(u)\,du.$$ Since all the integrands are positive, so are all the integrals, and it follows that

$$x^{-1}\varphi(x) - x^{-3}\varphi(x) < Q(x) < x^{-1}\varphi(x)$$

which is Eq. (4.4) of Ross. Clearly, the bounds tend to $-\infty$ and $+\infty$ as $x \to 0$ and thus are not useful for small x. On the other hand, the **ratio** of the upper and lower bounds on Q(x) is $1 + (x^2-1)^{-1}$ which approaches 1 as $x \to \infty$. Note that if either bound is used as an approximation for the value of Q(x), the maximum relative error is also **guaranteed** to decrease as x increases. For example, at x = 5, the maximum relative error is $1/24 \approx 4.2\%$.

(d) Since $t + x > t - x > 0$, $(t+x)(t-x) = t^2 - x^2 > (t-x)^2$. Hence, $\exp-(t^2-x^2) < \exp-(t-x)^2$ and therefore $$\exp\left(\frac{x^2}{2}\right)Q(x) = \int_x^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{t^2-x^2}{2}\right)dt < \int_x^{\infty}\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{(t-x)^2}{2}\right)dt = 1/2.$$ What? Why 1/2?

(e) The bound $(1/2)\exp(-x^2/2)$ is tighter for $x < \sqrt{2/\pi} \approx 0.7979$.

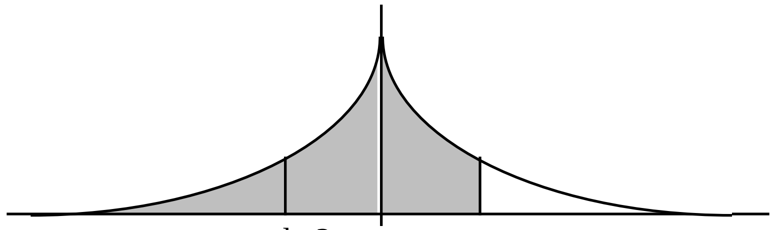
5. $Y(f) = H(f)X(f) = 2\bullet\exp(-\pi f^2)$ for $|f| \le 1$, and hence $y(0) = \int_{-\infty}^{\infty}Y(f)\bullet\exp(j2\pi f\bullet 0)df = \int_{-1}^{1}2\bullet\exp(-\pi f^2)df$

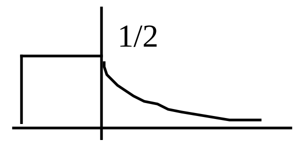
$$= 2\bullet\left[\Phi(\sqrt{2\pi}) - \Phi(-\sqrt{2\pi})\right] = 2\bullet\left[2\bullet\Phi(\sqrt{2\pi})-1\right] = 2\bullet[2\bullet\Phi(2.38\ldots) - 1] = 1.9652\ldots$$

6.(a) From the figure shown below, we see that the pdf is symmetric about u = 0. Hence, we get that

$$P\{\mathbf{X} \le \ln 2\} = \frac{1}{2} + \int_0^{\ln 2}\frac{1}{2}\exp(-u)\,du = \frac{1}{2} + \left[-\frac{1}{2}\exp(-u)\right]_0^{\ln 2} = \frac{3}{4}.$$ Notice that $P\{0 \le \mathbf{X} \le \ln 2\} = \frac{1}{4}$

(b) $$P\{|\mathbf{X}| \le \ln 2 \mid \mathbf{X} \le \ln 2\} = \frac{P\{|\mathbf{X}| \le \ln 2\} \cap \{\mathbf{X} \le \ln 2\}\}}{P\{\mathbf{X} \le \ln 2\}} = \frac{P\{|\mathbf{X}| \le \ln 2\}}{P\{\mathbf{X} \le \ln 2\}} = \frac{2P\{0 \le \mathbf{X} \le \ln 2\}}{3/4} = \frac{1/2}{3/4} = \frac{2}{3}.$$





(c) $\cos(\pi\mathbf{X}/2) = 0$ if $\mathbf{X}$ is an odd integer, and

$P\{\cos(\pi\mathbf{X}/2) < 0\} = \ldots + P\{-7 < \mathbf{X} < -5\} + P\{-3 < \mathbf{X} < -1\} + P\{1 < \mathbf{X} < 3\} + P\{5 < \mathbf{X} < 7\} + \ldots$

$= 2[P\{1 < \mathbf{X} < 3\} + P\{5 < \mathbf{X} < 7\} + \ldots]$ (by symmetry) $= [\exp(-1) - \exp(-3)] + [\exp(-5) - \exp(-7)] + \ldots$

$= \exp(-1)\bullet\dfrac{1}{1+\exp(-2)} = \dfrac{1}{2\bullet\cosh(1)} = 0.324\ldots$ on summing the geometric series.

(d) The minimum value of $\mathbf{I}$ is -1. $F_{\mathbf{I}}(b) = P\{\mathbf{I} \le b\} = 0$ if $b < -1$.

For $b \ge -1$, $F_{\mathbf{I}}(b) = P\{\mathbf{I} \le b\} = P\{e^{\mathbf{V}} - 1 \le b\} = P\{\mathbf{V} \le \ln(b+1)\} = F_{\mathbf{V}}(\ln(b+1))$. But,

$$F_{\mathbf{V}}(a) = \begin{cases} e^{a}/2, & a \le 0, \\ 1 - e^{-a}/2, & a > 0. \end{cases} \qquad \text{Thus, } F_{\mathbf{I}}(b) = \begin{cases} \dfrac{b+1}{2} & -1 \le b \le 0, \\ 1 - \dfrac{1}{2(b+1)}, & b > 0, \end{cases}$$

$$\text{and } f_{\mathbf{I}}(b) = \begin{cases} \dfrac{1}{2} & -1 \le b \le 0, \\ \dfrac{1}{2(b+1)^2}, & b > 0. \end{cases}$$

Note that the pdf has constant value 1/2 in the range $-1 \le b \le 0$.