1.(a) The maximum-likelihood estimate of p is the observed relative frequency, i.e. $\hat{p} = \mathbf{X}/N$.

(b) Since $E[\mathbf{X}] = Np$ and $\mathrm{var}(\mathbf{X}) = Np(1-p)$, the Chebyshev inequality gives

$$P\{|\hat{p} - p| \geq 0.02\} = P\{|\mathbf{X} - Np| > 0.02N\} \leq \frac{Np(1-p)}{0.0004N^2} = \frac{p(1-p)}{0.0004N} \leq \frac{1}{0.0016N}$$

where we have used the fact that the maximum value of $p(1-p)$ is 0.25 at $p = 0.5$. The right side of the above inequality must be no more than 0.05, and hence $N \geq 1/(0.0016 \bullet 0.05) = 12{,}500$. We conclude that polling 12,500 voters guarantees the desired confidence level regardless of the value of the parameter p.

2.(a) $\mathbf{X}$ is a binomial random variable with parameters (n, p). $E[\mathbf{X}] = np$. $\mathrm{var}(\mathbf{X}) = np(1-p)$.

(b) $P\{\mathbf{X} > n/2\} = \sum_{i=(n+1)/2}^{n} \binom{n}{i} p^i (1-p)^{n-i}$. For $n = 3$, $p = 10^{-2}$, this is $3p^2(1-p) + p^3 = 3p^2 - 2p^3 \approx 3 \times 10^{-4}$

Repetition *does* reduce the probability of error, but also reduces the data rate by a factor of 3 (n generally).

(c) $P\{\mathbf{X} > n/2\} = P\{(\mathbf{X}-np) > n/2 - np\} \leq P\{|\mathbf{X}-np| > n/2 - np\} \leq \frac{np(1-p)}{[n(1/2-p)]^2} = \frac{p(1-p)}{n(0.5-p)^2} = \frac{4.1232\times10^{-2}}{n}$

which is less than 10^{-5} for $n > 4123$.

3.(a) $P\{\mathbf{X} \leq 100\} = 1 - P\{\mathbf{X} > 100\} = 1 - [P\{\mathbf{X} = 105\}+P\{\mathbf{X} = 104\}+P\{\mathbf{X} = 103\}+P\{\mathbf{X} = 102\}+P\{\mathbf{X} = 101\}]$

$$= 1 - \left[(0.9)^{105} + \binom{105}{1}(0.9)^{104}(0.1) + \binom{105}{2}(0.9)^{103}(0.1)^2 + \binom{105}{3}(0.9)^{102}(0.1)^3 + \binom{105}{4}(0.9)^{101}(0.1)^4\right]$$

$= 0.983283...$

(b) If $\mathbf{X}$ is a binomial random variable with parameters (n, p), then $\mathbf{Y} = n - \mathbf{X}$ is a binomial random variable with parameters (n, 1–p). Thus, the number of no-shows is a binomial random variable with parameters (105, 0.1). Since n is large and p is small, it is reasonable to approximate this as a Poisson random variable with parameter $\lambda = np = 10.5$.

(c) $\{\mathbf{Y} \geq 5\} = 1 - P\{\mathbf{Y} < 5\} = 1 - [P\{\mathbf{Y} = 0\} + P\{\mathbf{Y} = 1\} + P\{\mathbf{Y} = 2\} + P\{\mathbf{Y} = 3\} + P\{\mathbf{Y} = 4\}]$

$= 1 - \exp(-10.5)[1 + (10.5) + (10.5)^2/2 + (10.5)^3/6 + (10.5)^4/24] = 0.978906\ldots$ which is not bad….

(d) Let $\mathbf{Y}$ denote the number of passengers who show up for the flight, regardless of where they came from. When the connecting flight is on time, 15 passengers are guaranteed to show up. Of the other 90, the number who show up is a binomial random variable $\mathbf{Z}$ with parameters (90, 0.9). Thus, $\mathbf{Y} = 15 + \mathbf{Z}$ when the connecting flight is on time. Otherwise, when the connecting flight is late, only $\mathbf{Z}$ passengers show up, and hence $\mathbf{Y} = \mathbf{Z}$ in this case.

Thus, $P\{\mathbf{Y} \leq 100 \mid \text{on time}\} = P\{\mathbf{Z} \leq 85\} = 1 - P\{\mathbf{Z} > 85\} = = 0.95345...$ and $P\{\mathbf{Y} \leq 100 \mid \text{late}\} = 1$.

Hence, $P(\mathbf{Y} \leq 100) = (0.95345...)\times(1/3) + 1\times(2/3) = 0.98448....$ Note that the only way for the airline to better the chances of everyone getting a seat is to increase the probability that the connecting flight is late!

4. P{all three coins show the same result} = P{hhh} + P{ttt} = 1/4. Thus,

(a) The number of rounds is a geometric random variable with parameter 3/4 and hence the average number of rounds is 4/3. Alternatively, the sum $(3/4) + 2\times(3/4)(1/4) + 3\times(3/4)(1/4)^2 + \ldots = (3/4)\times1/[1 - 1/4]^2 = 4/3$

(b) $P\{\text{no more than 5 rounds} \mid \text{at least three rounds}\} = \frac{P\{\text{at least 3 and no more than 5}\}}{P\{\text{at least three}\}} = \frac{q^2p + q^3p + q^4p}{q^2}$

$= p(1 + q + q^2) = 63/128.$

(c) Now, P{all three coins show the same result} = P{hhh} = P{hh} = 1/4. Note that there are only four possible outcomes now, viz. hhh, hht, hth, htt. Hence, we get the same answers as in parts (a) and (b)!

5.(a) Maximize your chances of living by predicting the most probable outcome. According to Proposition 6.1, p.145 in Ross, 6th ed.) the most probable number of Heads is $\lfloor (N+1)p \rfloor = 110$, so predict that 110 Heads will occur. This has probability $\binom{1000}{110}(0.11)^{110}(0.89)^{890} = 0.04028\ldots$ Not great odds of survival…

It is important to understand that the answer to this problem is *not* the mean value Np, even though both formulas happen to give the same number in this particular instance.

(b) P(Heads occuring for first time on k-th toss) $= (1-p)^{k-1}p$ is a *decreasing* function of k. So predict that Heads occurs for the first time on the first toss, and survive with probability 0.11! Note that if you had chosen the mean value $1/p = 9.09\ldots$ and rounded it off to 9, your chances of surviving would be only $(0.89)^8 0.11 = 0.0433\ldots$ This is yet another illustration of the central idea: you have the best chance of survival if you predict that the most likely event will occur, and not if you predict that the "average" event will occur, i.e. use probabilistic notions correctly in order to live long and prosper!

(c) P(105th head on kth toss) $= \binom{k-1}{104} p^{105} q^{k-105}$. To find which value of k maximizes this, consider the ratio

$\frac{P(\text{105th head on kth toss})}{P(\text{105th head on (k–1)th toss})} = \frac{k-1}{k-105} q \geq 1$ as long as $k \leq \frac{105-q}{1-q} = 946.45\ldots$ So predict that the 105th head will occur on the 946th toss. The probability of this event is $\binom{945}{104}p^{105}q^{841} = 0.00455\ldots$ Ouch!

(d) P(Heads occurring for first time on 12th toss) = $(1–p)^{11}p$. It is easy to show that this probability is maximum at $p^* = 1/12 = 0.08333\ldots$

(e) The maximum-likelihood estimate of p is the number p* that maximizes $\binom{993}{299}p^{300}(1–p)^{694}$. The usual calculus method shows that the maximum is at $p^* = 300/994 = 0.30181086519\ldots$

(f) The maximum-likelihood estimate of the unknown value of p is the number p* that maximizes $\binom{1000}{300}p^{300}(1–p)^{700}$. With the same method as before, the maximum is at $p^* = 300/1000 = 0.3$

(g) The difference in the two cases is that after having observed the 300th head on the 994th toss, we can estimate p* as 300/994 = 0.3018…*without knowing what happened on the next 6 trials.* If we are told that Tails occurred on the next 6 trials, we *do* receive additional information which causes us to be pessimistic about the value of p, and hence we revise the estimated value of p *downwards* to 300/1000 = 0.3.

(h) Since p = 0.11, the coin comes up Tails very frequently, probably because the weight in the bone-headed king's image makes the Heads side be underneath more often. The father probably was not quite so dense.

6.(a) $P(A|B^c) = 1 - P(A^c|B^c) = 0.6$. $P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = 0.39$.
$P(B|A) = P(AB)/P(A) = P(A|B)P(B)/P(A) = 0.21/0.39 = 7/13$.

(b) $P(F) = P(EF)/P(E|F) = P(F|E)P(E)/P(E|F) = (1/2) – (1/4)/(1/3) = 3/8$.

(c) 1 ≥ P(G H) = P(G) + P(H) – P(GH).
Hence, $P(GH) \geq 2/3 + 2/3 – 1 = 1/3$, and $P(G|H) = P(GH)/P(H) = P(GH)/(2/3) \geq (1/3)/(2/3) = 1/2$.

7.(a) $P(R_1) = 6/10$, $P(R_2|R_1) = 5/9$, $P(R_2|R_1^c) = 6/9$. Geez, that was an easy one!

(b) $P(R_2) = P(R_2|R_1)P(R_1) + P(R_2|R_1^c)P(R_1^c) = (5/9)\bullet(6/10) + (6/9)\bullet(4/10) = 54/90 = 6/10$. Surprised?

(c) $P(R_1) = 6/10$ as before; $P(R_2|R_1) = 9/13$, $P(R_2|R_1^c) = 6/13$.

$P(R_2) = P(R_2|R_1)P(R_1)+P(R_2|R_1^c)P(R_1^c) = (9/13)\bullet(6/10) + (6/13)\bullet(4/10) = 78/130 = 6/10$. Surprised **now**?
A ball is being picked at random from an urn with 13 balls in it. It would seem that **all** probabilities would be of the form k/13, but instead we get an answer of 6/10. This is called Polya's urn scheme. Note that the answer is 6/10 if *any* number of additional balls of the same color are added; the result is not a magical property of 3. Furthermore, if the second ball is returned to the urn with 3 additional balls of the same color, the probability of the third ball being drawn being red is *still* 6/10. Weird, isn't it?

Noncredit Exercises:

1. $(d/d\lambda)P\{\mathbf{X} = k\} = (d/d\lambda)\exp(–\lambda)\lambda^k/k! = \exp(–\lambda)[k\lambda^{k−1}/k! – \lambda^k/k!\ ] = 0$ at $\lambda = k$. Since $\exp(–\lambda)\lambda^k/k! > 0$ for all λ (0,∞) and has value 0 at λ = 0 and also at λ = ∞, λ = k is a maximum. Thus the maximum–likelihood estimate of λ is $\hat{\lambda} = k$. Since $\lambda \approx Np$, this is consistent with the binomial estimate $\hat{p} = k/N$.

2.(a) $\sum_{k\ \text{even}} P\{\mathbf{X} = k\} = \exp(–\lambda)\sum_{k=0}^{\infty}\lambda^{2k}/2k! = \exp(–\lambda)\cosh(\lambda)$ via the known series for cosh. Geez, that was easy!

(b) $[1+(1 – 2p)^n]/2 = [1+(1 – 2np/n)^n]/2 \approx [1 + \exp(–2np)]/2 = \exp(–np)[\exp(np) + \exp(–np)]/2$
$= \exp(–np)\cosh(np) = \exp(–\lambda)\cosh(\lambda)$ on setting $\lambda = np$. He's losing his touch; that was even easier!

3. Let A and B denote respectively the events that your **first** choice and your **final** choice is the curtain concealing the prize. $P(A) = 1/3$, $P(A^c) = 2/3$.

(a) If you always switch, $P(B|A) = 0$, while $P(B|A^c) = 1$. Hence, $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 2/3$.

(b) If you never switch, then $P(B|A) = 1$, while $P(B|A^c) = 0$. Hence, $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 1/3$.

(c) If you decide at random, then $P(B|A) = P(B|A^c) = 1/2$ and $P(B) = P(B|A)P(A) + P(B|A^c)P(A^c) = 1/2$ also.
Monty is correct in his assertion. (Would he lie to you? Besides, you saw it on **TV**, so it must be true!!!!)

4. This game is different from the one in Problem 3 in that *you have no idea what the rules of the game are.* If the man is intent on separating you from your money as quickly as possible, he will not offer the chance to switch unless you picked the shell hiding the pea in the first place! That is, if you picked the wrong shell, the man will reveal the pea and you will lose your bet. Of course, if you look like a person willing to play several rounds, the man may set you up by playing by Monty's rules (and allowing you to win with probability 2/3) for some time. Then you will place a large bet on the wrong shell, and all of a sudden, you will not be given the choice of changing your bet! Personally, I would stick with the shell originally chosen since it gives me at least a 1/3 probability of winning regardless of the man's strategy; your experience (and monetary losses) may vary