ECE 313 FINAL EXAMINATION

Friday December 14, 2001

Three hours

1. Check the appropriate box for each of the statements below. No justification is required, but, in order to discourage guessing, your score will be reduced by 3 points for each wrong answer (You get +3 points for each right answer; 0 for no answer).

(a) Let $\mathbf{X}$ denote a continuous random variable whose pdf $f_{\mathbf{X}}(u)$ is an *even function* of u, i.e. $f_{\mathbf{X}}(u) = f_{\mathbf{X}}(-u)$. Let $F_{\mathbf{X}}(u)$ denote the CDF of $\mathbf{X}$, and let $\alpha > 0$.

TRUE	FALSE	
n	n	$F_{\mathbf{X}}(\alpha) = F_{\mathbf{X}}(-\alpha)$
n	n	$P\{\mathbf{X} > \alpha\} = F_{\mathbf{X}}(-\alpha)$
n	n	$P\{\div\mathbf{X}\div < \alpha\} = 2F_{\mathbf{X}}(\alpha) - 1$
n	n	$P\{\div\mathbf{X}\div > \alpha\} = 2F_{\mathbf{X}}(\alpha)$
n	n	$P\{\div\mathbf{X}\div > \alpha\} = 2[1 - F_{\mathbf{X}}(\alpha)]$

(b) $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{Z}$ denote continuous random variables with finite variance. Suppose that $\mathbf{Y} = -\mathbf{X}$, and $\mathbf{Z} = 2\mathbf{X}$.

TRUE	FALSE	
n	n	$\rho_{\mathbf{X},\mathbf{Y}} = -1$
n	n	$\rho_{\mathbf{X},\mathbf{Z}} = +2$
n	n	$f_{\mathbf{Y}}(u) = f_{\mathbf{X}}(-u)$
n	n	$f_{\mathbf{Z}}(u) = (1/2)f_{\mathbf{X}}(u/2)$
n	n	Since $\mathbf{Z} = \mathbf{X} + \mathbf{X}$, $f_{\mathbf{Z}} = f_{\mathbf{X}} * f_{\mathbf{X}}$
n	n	$F_{\mathbf{Y}}(u) = F_{\mathbf{X}}(-u)$
n	n	$F_{\mathbf{Z}}(u) = (1/2)F_{\mathbf{X}}(u/2)$

2. Check **one box** for each of the following statements. No justification is required, but, in order to discourage guessing, your score will be reduced by 2 points for a wrong answer (You get +8 points for right answer; 0 for no answer).

(a) Given n random variables $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$,

$E[\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n] = E[\mathbf{X}_1] + E[\mathbf{X}_2] + \ldots + E[\mathbf{X}_n]$

n **only if** the random variables are *independent.*

n **only if** the random variables are *uncorrelated.*

n **only if** the random variables are *jointly Gaussian.*

n **only if** the random variables have *zero means.*

n **always**.

(b) Given n random variables $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$,

$\text{var}[\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n] = \text{var}[\mathbf{X}_1] + \text{var}[\mathbf{X}_2] + \ldots + \text{var}[\mathbf{X}_n]$

n **if** the random variables have *identical marginal distributions.*

n **if** the random variables are *uncorrelated.*

n **if** the random variables are *jointly Gaussian.*

n **if** the random variables have *zero means.*

n **always**.

3. Each box of Cornies, the breakfast of silver medalists, contains one picture, which is equally likely to be a picture of Luke Skywalker, Darth Vader, or Jabba the Hut, independently of which picture is in any other box of Cornies. Little Jimmy Kirk of Cedar Rapids, Iowa, asks his mother to buy boxes of Cornies until he has at least one picture of each of these three beings, and his doting mother agrees to do so.

(a) What is the minimum number of boxes of Cornies that Mrs Kirk must buy?

(b) Let **X** denote the number of boxes of Cornies Mrs Kirk purchases until such time as Jimmy has acquired at least one picture of each of the three entities. What is P{**X** > 3}?

(c) More generally, what is P{**X** > n} for n ≥ 3 ?

(d) What is P{**X** > n} for n < 3 ?

(e) What is the expected number of boxes of Cornies purchased by Mrs Kirk?
Note: You can use any of the several different ways of computing the desired answer. However, since **X** takes on (nonnegative) integer values only, you might find it convenient to obtain the answer from the answers to parts (c) and (d) via the formula: $E[\mathbf{X}] = \sum_{n=0}^{\infty} P\{\mathbf{X} > n\}$

4. Let **X** denote the time of the first arrival after t = 0 in a Poisson process with arrival rate λ.

(a) **(6 points)** Fill in the blanks in the statement below. No work needs to be shown.

X is [□ a □ an] random variable with [□ parameter □ parameters] and for T > 0, [$F_{\mathbf{X}}(T) = P\{0 < \mathbf{X} \le T\} =$]

(b) Let A denote the event that there is exactly one arrival in the interval (0,T]. What is P(A)?

(c) Is the P(A) that you found for part (b) the same as the value of $F_{\mathbf{X}}(T)$ that you gave in part (a) ? Explain why the two are the same (or are different, as appropriate)

(d) For 0 < τ < T, what is the *conditional* probability that {**X** ≤ τ} *given* the event A, that is, given that there was exactly one arrival in (0, T]?

(e) Find the *conditional* pdf of **X** *given* the event A

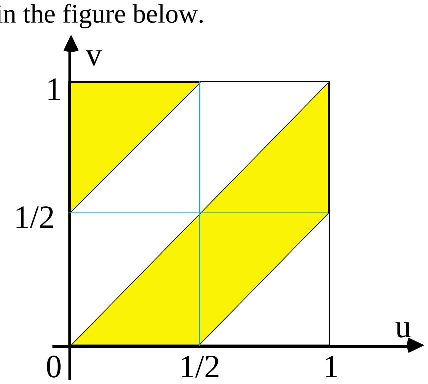
5. **B** and **C** are random variables with joint pdf

$$f_{\mathbf{B},\mathbf{C}}(u,v) = \begin{cases} u + v, & 0 < u < 1,\ 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) TRUE or FALSE? **B** and **C** are independent random variables. Explain your answer.

(b) What is the probability that the roots of the polynomial $x^2 + 2\mathbf{B}x + \mathbf{C}$ are real?

6. The joint probability density function $f_{\mathbf{X},\mathbf{Y}}(u,v)$ for the continuous random variables **X** and **Y** has constant value 2 on the shaded region in the figure below.



(a) Find $f_{\mathbf{X}}(u)$, the marginal probability density function for **X**. In order to obtain full credit, you must specify the value of $f_{\mathbf{X}}(u)$ for all real numbers u.

(b) Compute P{**X** > **Y**}.

(c) Are **X** and **Y** independent ? Explain.

(d) Find P{**X** + **Y** < 1/3}, P{**X** + **Y** < 2/3}, P{**X** + **Y** > 4/3} and P{**X** + **Y** > 5/3}.

(e) Find the pdf of the random variable **Z** = **X** + **Y**.

7. The jointly Gaussian random variables **X** and **Y** have means 0 and 7 respectively, variances 4 and 16 respectively, and correlation coefficient 1/16.

(a) Find the probability density function of **Z** = 5**X** + **Y**. In order to obtain full credit, you must specify the value of $f_{\mathbf{Z}}(w)$ for all real numbers w.

(b) Find the numerical value of P{**Y** > 3**X**}.