University | **Solutions to Second Hour Exam** | **ECE 313**

of Illinois | | **Fall 2001**

1. (a) ‹ $P\{\mathbf{X} > b\} = 1 - F_{\mathbf{X}}(b)$. ` If $F_{\mathbf{X}}(a) < F_{\mathbf{X}}(b)$, then $a < b$.

´ If $a < b$, then $F_{\mathbf{X}}(a) < F_{\mathbf{X}}(b)$. ˆ $F_{\mathbf{X}}(u) = 1/2$ for some u, $-\infty < u < \infty$.

The first two properties hold for all CDFs since $F_{\mathbf{X}}(u)$ is a *right-continuous non-decreasing* function. But, if $a < b$, it is possible that $F_{\mathbf{X}}(a) = F_{\mathbf{X}}(b)$ because the CDF did not increase between those points (it cannot decrease, of course.) As a counterexample for ˆ , note that for a Bernoulli random variable with parameter $p \neq 1/2$, $F_{\mathbf{X}}(u)$ takes on values 0, p, and 1 only. Thus, ´ and ˆ are not properties of all CDFS.

(b) ‹ $f_{\mathbf{X}}(u) \leq 1$ for all u, $-\infty < u < \infty$. ` $\lim_{u\to -\infty} f_{\mathbf{X}}(u) = 0$

´ $\lim_{u\to \infty} f_{\mathbf{X}}(u) = 1$ ˆ $P\{a < \mathbf{X} < b\} = P\{a \leq \mathbf{X} \leq b\}$

The second and fourth properties are satisfied by all pdfs. The first need not hold: the value of the function is a probability *density*, not a probability mass, and can exceed 1 for some pdfs. On the other hand, **no** pdf can possibly satisfy the third property.

2. $P\{\{|\mathbf{X} - 4| > 3\} = P\{\mathbf{X} > 7\} + P\{\mathbf{X} < 1\} = 1 - \Phi\left(\frac{7-2}{5}\right) + \Phi\left(\frac{1-2}{5}\right) = 1 - \Phi(1) + \Phi(-0.2)$

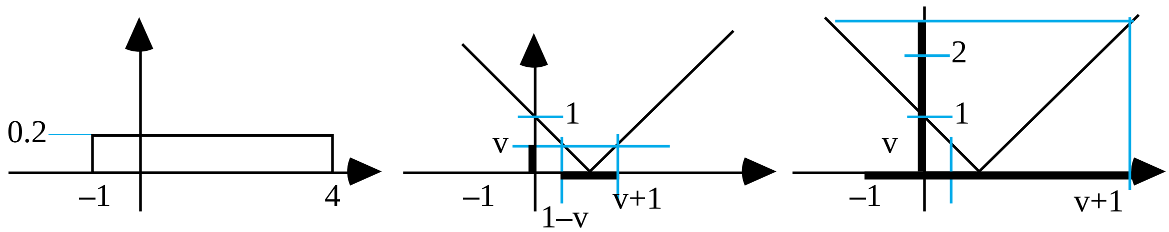
$= 1 - \Phi(1) + 1 - \Phi(0.2) = 2 - 0.8413 - 0.5793 = 0.5794.$

$P\{\mathbf{X} < 3 \mid \mathbf{X} > 2\} = P\{2 < \mathbf{X} < 3\}/P\{\mathbf{X} > 2\} = 2\left[\Phi\left(\frac{3-2}{5}\right) - \Phi\left(\frac{2-2}{5}\right)\right] = 2(\Phi(0.2) - 0.5)$

$= 1.1586 - 1 = 0.1586$

3.(a) The pdf of **X** has value 0.2 for $-1 \leq u \leq 4$, is as shown below in the left-hand figure below. We have that

$$E[\mathbf{Y}] = E[|\mathbf{X}-1|] = \int_{-1}^{1} 0.2(1-u)\,du + \int_{1}^{4} 0.2(u-1)\,du = 0.2(u - u^2/2)\Big|_{-1}^{1} + 0.2(u^2/2-u)\Big|_{1}^{4} = 1.3.$$



(b) **Y** takes on values in the range [0, 3]. From the middle and right-hand figures above, we see that for any v, $0 \leq v \leq 2$, $F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \leq v\} = P\{1-v \leq \mathbf{X} \leq v+1\} = 0.2(v+1-(1-v)) = 0.4v$, while for any v, $2 \leq v \leq 3$, $F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \leq v+1\} = P\{\mathbf{X} \leq v\} = 0.2(v+1-(-1)) = 0.2(v+2)$.

Hence, $f_{\mathbf{Y}}(v) = \begin{cases} 0.4, & 0 \leq v \leq 2, \\ 0.2, & 2 \leq v \leq 3 \\ 0, & \text{elsewhere} \end{cases}$ which is easily verified to be a valid pdf.

4.(a) $\mathbf{N}(3, 6]$ is a Poisson random variable with parameter 3µ. Hence, $P\{\mathbf{N}(3, 6] = 2\} = \exp(-3\mu)\bullet(3\mu)^2/2!$.

(b) If there are 2 arrivals in (0, 6], and also 2 arrivals in (3, 9], then it must be that in the *disjoint* intervals (0, 3], (3, 6], and (6, 9], there are (2, 0 ,2) arrivals respectively, or (1, 1, 1) arrivals respectively, or (0, 2, 0) arrivals respectively, which correspond to values of $\mathbf{N}(3, 6]$ of 0, 1, and 2 respectively. Hence, by the independence of arrivals in disjoint intervals, we get that

P(AB) = $P\{\mathbf{N}(3,6] = 0\}\bullet P\{$two arrivals in (0, 3] and two arrivals in (6, 9]$\}$

+ $P\{\mathbf{N}(3,6] = 1\}\bullet P\{$one arrival in (0, 3] and one arrival in (6, 9]$\}$

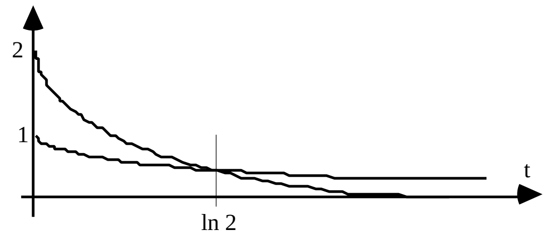
+ $P\{\mathbf{N}(3,6] = 2\}\bullet P\{$no arrivals in (0, 3] and no arrivals in (6, 9]$\}$

$= \exp(-3\mu)\bullet[\exp(-3\mu)\bullet(3\mu)^2/2!]^2 + [\exp(-3\mu)\bullet(3\mu)]\bullet[\exp(-3\mu)\bullet(3\mu)]^2 + [\exp(-3\mu)\bullet(3\mu)^2/2!]\bullet[\exp(-3\mu)]^2$

$= \exp(-9\mu)\bullet[81\mu^4/4 + 27\mu^3 + 9\mu^2/2]$. Hence, we have

$$P\{\mathbf{N}(3, 6] = 1 \mid AB\} = P\{\{\mathbf{N}(3, 6] = 1\} \cap AB\}/P(AB) = \frac{27\mu^3}{81\mu^4/4 + 27\mu^3 + 9\mu^2/2} = \frac{12\mu}{9\mu^2 + 12\mu + 2}$$

5.(a) **X** is an exponential random variable with parameter $\lambda_0 = 1$ or $\lambda_1 = 2$ according as hypothesis H_0 or H_1 is true. Thus, the pdfs $f_0(u)$ and $f_1(u)$ are as shown below where the crossing-point is easily found by equating $2\bullet\exp(-2t) = \exp(-t)$ and solving to get $t = \ln 2$. Thus, we see that the maximum-likelihood decision rule chooses H_1 if $T < \theta = \ln 2$, and H_0 if $T > \ln 2$. Hence, $i = 0$, $\theta = \ln 2$ in the decision rule exhibited on the exam..



(b) $P_{FA} = P\{\mathbf{X} < \ln 2 \mid H_0 \text{ is true}\} = 1 - P\{\mathbf{X} > \ln 2 \mid H_0 \text{ is true}\} = 1 - \exp(-\lambda_0 \bullet \ln 2) = 1 - \exp(-\ln 2) = 1/2$.
$P_{MD} = P\{\mathbf{X} > \ln 2 \mid H_1 \text{ is true}\} = \exp(-\lambda_1 \bullet \ln 2) = \exp(-2 \bullet \ln 2) = \exp(-4)$.

(c) The MEP rule compares $\pi_1 \bullet f_1(T)/\pi_0 f_0(T) = (0.4)\bullet 2\bullet\exp(-2T)/[(0.6)\bullet\exp(-T)] = (4/3)\bullet\exp(-T)$ to the threshold 1 and chooses H_1 if this ratio exceeds 1. Thus, the MEP decision rule chooses H_1 if $\exp(T) < 4/3$, i.e. if $T < \ln(4/3)$. Note that $\ln(4/3) < \ln 2$ and the MEP rule is thus playing the odds and choosing the more likely hypothesis H_0 whenever $\ln(4/3) < T < \ln 2$, in which cases the maximum-likelihood rule would be choosing H_1. We have that i = 0 as before, and $\theta = \ln(4/3)$.

(d) More generally, the MEP rule chooses H_1 if $\exp(T) < 2\pi_1/\pi_0 = 2(1-\pi_0)/\pi_0$ and H_0 if $\exp(T) > 2\pi_1/\pi_0$. If $\pi_0 > 2/3$, then $2(1-\pi_0)/\pi_0 < 1$, and hence $\exp(T) > 2\pi_1/\pi_0$ for all $T > 0$. It follows that if $\pi_0 > 2/3$, then the MEP rule always chooses H_0. On the other hand, $2\pi_1/\pi_0 = 2(1-\pi_0)/\pi_0$ is finite for all $\pi_0 > 0$, and hence there is always a T for which $\exp(T) > 2\pi_1/\pi_0$. Thus, for any $\pi_0 > 0$, the MEP rule *will* choose H_0 on those trials on which the outcome exceeds $\ln(2\pi_1/\pi_0)$, and will not always choose H_1. The MEP rule will always choose H_1 only when $\pi_0 = 0$, in which case, it should be intuitively obvious that it makes no sense for an MEP rule to make any choice other than H_1!!