1. (a) ① \( P(X > b) = 1 - F_X(b) \).
     ② If \( F_X(a) < F_X(b) \), then \( a < b \).
     ③ If \( a < b \), then \( F_X(a) < F_X(b) \).
     ④ \( F_X(u) = 0 \) for some \( u, -\infty < u < \infty \).

The first two properties hold for all CDFs since \( F_X(u) \) is a right-continuous non-decreasing function. But, if \( a < b \), it is possible that \( F_X(a) = F_X(b) \) because the CDF did not increase between those points (it cannot decrease, of course.) As a counterexample for ③, note that for a Bernoulli random variable with parameter \( p \neq 1/2 \), \( F_X(u) \) takes on values 0, 0.5, and 1 only. Thus, ③ and ④ are not properties of all CDFs.

(b) ① \( f_X(u) \leq 1 \) for all \( u, -\infty < u < \infty \).
     ② \( \lim_{u \to -\infty} f_X(u) = 0 \).
     ③ \( \lim_{u \to \infty} f_X(u) = 1 \).

The second and fourth properties are satisfied by all pdfs. The first need not hold: the value of the function is a probability density, not a probability mass, and can exceed 1 for some pdfs. On the other hand, no pdf can possibly satisfy the third property.

2. \( P(|X - 4| > 3) = P(X > 7) + P(X < 1) = 1 - \Phi(\frac{7-2}{5}) + \Phi(\frac{1-2}{5}) = 1 - \Phi(1) + \Phi(-0.2) = 1 - 0.8413 - 0.5793 = 0.5794. \)

\( P(X < 3 | X > 2) = \frac{P(2 < X < 3)}{P(X > 2)} = \frac{2(\Phi(\frac{3-2}{5}) - \Phi(\frac{2-2}{5}))}{2(\Phi(0.2) - 0.5)} = \frac{0.5794}{1.1586} = 0.4966. \)

3. (a) The pdf of \( X \) has value 0.2 for \(-1 \leq u \leq 4 \), as shown below in the left-hand figure below. We have that

\[ E[Y] = E[|X-1|] = \left\{ \begin{array}{ll}
0.2(1-u) \text{ for } & -1 \leq u \leq 2, \\
0.2(u-1) \text{ for } & 3 \leq u \leq 4,
\end{array} \right. \]

\[ 0.2(1-u) du + 0.2(u-1) du = 0.2(u^2/2) \bigg|_{-1}^{4} + 0.2(u^2/2-u) \bigg|_{1}^{4} = 1.3. \]

(b) \( Y \) takes on values in the range \([0, 3]\). From the middle and right-hand figures above, we see that for any \( v, \ 0 \leq v \leq 2 \), \( F_Y(v) = P(Y \leq v) = P[1-v \leq X \leq v+1] = 0.2(v+1-(1-v)) = 0.4v \), while for any \( v, \ 2 \leq v \leq 3 \), \( F_Y(v) = P(Y \leq v+1) = P[X \leq v] = 0.2v+1-(1-v) = 0.2v+2 \).

Hence, \( f_Y(v) = \left\{ \begin{array}{ll}
0.4, & 0 \leq v \leq 2, \\
0.2, & 2 \leq v \leq 3, \\
0, & \text{ elsewhere,}
\end{array} \right. \) which is easily verified to be a valid pdf.

4. (a) \( N(3, 6) \) is a Poisson random variable with parameter \( 3\mu \). Hence, \( P[N(3, 6) = 2] = \exp(-3\mu)(3\mu)^2/2! \).

(b) If there are 2 arrivals in (0, 6], and also 2 arrivals in (3, 9], then it must be that in the disjoint intervals (0, 3], (3, 6], and (6, 9], there are (2, 0, 2) arrivals respectively, or (1, 1, 1) arrivals respectively, or (0, 2, 0) arrivals respectively, which correspond to values of \( N(3, 6) \) of 0, 1, and 2 respectively.

Hence, by the independence of arrivals in disjoint intervals, we get that

\[ P(AB) = P[N(3, 6) = 0] \cdot P[\text{two arrivals in (0, 3] and two arrivals in (6, 9]} + P[N(3, 6) = 1] \cdot P[\text{one arrival in (0, 3] and one arrival in (6, 9]} + P[N(3, 6) = 2] \cdot P[\text{no arrivals in (0, 3] and no arrivals in (6, 9].} \]

\[ = \exp(-3\mu)\exp(-3\mu)(3\mu)^2/2! + \exp(-3\mu)(3\mu)^2/2! + \exp(-3\mu)(3\mu)^2/2! \cdot \exp(-3\mu)^2 \]

\[ = \exp(-9\mu)(81\mu^4/4 + 27\mu^3 + 9\mu^2/2). \]

Hence, we have

\[ P[N(3, 6) = 1 | AB] = P[N(3, 6) = 1 \cap AB]/P(AB) = \frac{27\mu^3}{81\mu^4/4 + 27\mu^3 + 9\mu^2/2} = \frac{12\mu}{9\mu^2 + 12\mu + 2}. \]

5. (a) \( X \) is an exponential random variable with parameter \( \lambda_0 = 1 \) or \( \lambda_1 = 2 \) according as hypothesis \( H_0 \) or \( H_1 \) is true. Thus, the pdfs \( f_0(u) \) and \( f_1(u) \) are as shown below where the crossing-point is easily found by equating \( 2\exp(-2t) = \exp(-t) \) and solving to get \( t = \ln 2 \). Thus, we see that the maximum-likelihood decision rule chooses \( H_1 \) if \( T < \theta = \ln 2 \), and \( H_0 \) if \( T > \ln 2 \). Hence, \( i = 0, \theta = \ln 2 \) in the decision rule exhibited on the exam.
(b) \[ P_{FA} = P(X < \ln 2 \mid H_0 \text{ is true}) = 1 - P(X > \ln 2 \mid H_0 \text{ is true}) = 1 - \exp(-\lambda_0 \cdot \ln 2) = 1 - \exp(-\ln 2) = 1/2. \]
\[ P_{MD} = P(X > \ln 2 \mid H_1 \text{ is true}) = \exp(-\lambda_1 \cdot \ln 2) = \exp(-2 \cdot \ln 2) = \exp(-4). \]

(c) The MEP rule compares \[ \pi_1 \cdot f_1(T) / \pi_0 \cdot f_0(T) = (0.4) \cdot 2 \cdot \exp(-2T) / [(0.6) \cdot \exp(-T)] = (4/3) \cdot \exp(-T) \] to the threshold 1 and chooses \( H_1 \) if this ratio exceeds 1. Thus, the MEP decision rule chooses \( H_1 \) if \( \exp(T) < 4/3 \), i.e. if \( T < \ln(4/3) \). Note that \( \ln(4/3) < \ln 2 \) and the MEP rule is thus playing the odds and choosing the more likely hypothesis \( H_0 \) whenever \( \ln(4/3) < T < \ln 2 \), in which cases the maximum-likelihood rule would be choosing \( H_1 \). We have that \( i = 0 \) as before, and \( \theta = \ln(4/3) \).

(d) More generally, the MEP rule chooses \( H_1 \) if \( \exp(T) < 2\pi_1 / \pi_0 = 2(1 - \pi_0) / \pi_0 \) and \( H_0 \) if \( \exp(T) > 2\pi_1 / \pi_0 \).

If \( \pi_0 > 2/3 \), then \( 2(1 - \pi_0) / \pi_0 < 1 \), and hence \( \exp(T) > 2\pi_1 / \pi_0 \) for all \( T > 0 \). It follows that if \( \pi_0 > 2/3 \), then the MEP rule always chooses \( H_0 \). On the other hand, \( 2\pi_1 / \pi_0 = 2(1 - \pi_0) / \pi_0 \) is finite for all \( \pi_0 > 0 \), and hence there is always a \( T \) for which \( \exp(T) > 2\pi_1 / \pi_0 \). Thus, for any \( \pi_0 > 0 \), the MEP rule will choose \( H_0 \) on those trials on which the outcome exceeds \( \ln(2\pi_1 / \pi_0) \), and will not always choose \( H_1 \). The MEP rule will always choose \( H_1 \) only when \( \pi_0 = 0 \), in which case, it should be intuitively obvious that it makes no sense for an MEP rule to make any choice other than \( H_1 \)!!