ECE 313

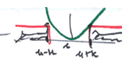
Probability with Engineering Applications

Lecture 41 – Limit Theorems II

Professor Seth Hutchinson
for Dilip V. Sarwate
Department of Electrical and Computer Engineering

Chebyshev's Inequality

Chebyshev's Inequality



$$P\{|X-\mu| \geq k\} \leq \frac{\sigma^2}{k^2}$$

$$P\{|X-\mu| \geq k\} = \int_{-\infty}^{\mu-k} f_X(u)\,du + \int_{\mu+k}^{\infty} f_X(u)\,du = \int_{-\infty}^{\infty} g(u) f_X(u)\,du$$

$$g(u) = \begin{cases} 1: & u \leq \mu-k \text{ or } u \geq \mu+k \\ 0: & \text{else} \end{cases} \Rightarrow h(u) = \frac{(u-\mu)^2}{k^2} \geq g(u)$$

$$\underbrace{P\{|X-\mu| \geq k\}}_{} = \int_{-\infty}^{\infty} g(u) f_X(u)\,du \leq \int_{-\infty}^{\infty} \frac{(u-\mu)^2}{k^2} f_X(u)\,du = \frac{\sigma^2}{k^2}$$

Weak Law of Large Numbers

Weak Law of Large Numbers

Thm. Let $X_1, X_2 \cdots$ be a sequence of i.i.d. rv's with $E\{X_i\} = \mu < \infty$, Then for ~~all~~ any $k > 0$

$$P\left\{\left|\frac{X_1+X_2\cdots X_n}{n} - \mu\right| \geq k\right\} \to 0 \text{ as } n \to \infty.$$

Pf. $Z = \frac{X_1 + \cdots X_n}{n}$, $\mu_Z = \frac{n\mu}{n}$, $Var(Z) = \frac{1}{n^2} n\sigma^2 = \frac{\sigma^2}{n}$

$$P\{|Z-\mu_Z| \geq k\} \leq \frac{\sigma_Z^2}{k^2} = \left(\frac{\sigma_X^2}{n} \cdot \frac{1}{k^2}\right)$$

Chebyshev, WLLN, Rel. Freq.

Chebyshev, WLLN, Rel. Freq:

Consider n indep. trials of some experiment, prob of success on any trial is p, $X_i = 1$ if success on ith trial, else $X_i = 0$.

$\mu_{X_i} = p$, $Var(X_i) = p(1-p)$ [$X_i \sim$ Bern. w/ p.]

$Z = X_1 + X_2 \cdots X_n$ = Total # successes

Z/n = rel. freq. of success

$E[Z/n] = p$

$Var(Z/n) = \frac{p(1-p)}{n}$

$$P\left\{\left|\frac{X_1+\cdots X_n}{n} - \mu\right| \geq \alpha\right\} = \underbrace{P\{|Z/n - p| \geq \alpha\}}_{} \leq \frac{p(1-p)}{n\alpha^2}$$

Prob {rel. freq. differs from true prob. of success by more than α}

Central Limit Theorem

Central Limit Theorem

Thm Let $X_1, X_2 \ldots$ be a sequence of i.i.d. rv's with $E\{X_i\} = \mu$, $Var(X_i) = \sigma^2$, then

$$Z = \frac{X_1 + X_2 \cdots X_n - n\mu}{\sigma\sqrt{n}}$$

tends to unit Gauss. as $n \to \infty$, i.e.

$$P\{Z \leq a\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2}\,du = \Phi(a).$$

$Y = \Sigma X_i \Rightarrow \mu_Y = n\mu$, $Var(Y) = n\sigma^2 \Rightarrow \sigma_Y = \sqrt{n}\sigma$

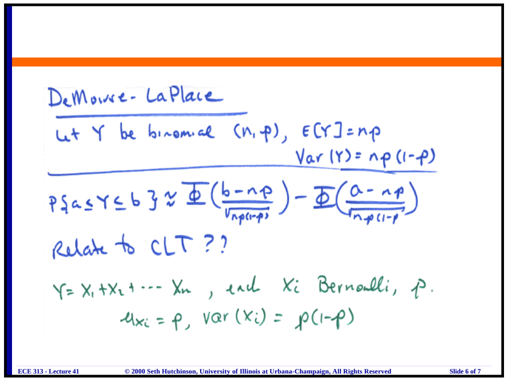
Example

Example

X_i are i.i.d, $E[X_i] = \mu$, $Var(X_i) = \sigma^2$, $i = 1 \ldots n$

Let $Y = \Sigma X_i$

$$P\{a \leq Y \leq b\} \approx \Phi\left(\frac{b - n\mu_X}{\sigma_X\sqrt{n}}\right) - \Phi\left(\frac{a - n\mu_X}{\sigma_X\sqrt{n}}\right)$$


DeMoivre-LaPlace
Let Y be binomial (n,p), $E[Y]=np$
$Var(Y)=np(1-p)$
$P\{a\leq Y\leq b\}\approx\Phi\left(\frac{b-np}{\sqrt{np(1-p)}}\right)-\Phi\left(\frac{a-np}{\sqrt{np(1-p)}}\right)$
Relate to CLT ??
$Y=X_1+X_2+\cdots X_n$, each X_i Bernoulli, p.
$\mu_{X_i}=p$, $var(X_i)=p(1-p)$
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