ECE 313
Probability with Engineering Applications
Lecture 40 – Limit Theorems

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Prediction

Prediction

Determine $g(X)$ to predict Y.

$E\{(Y-g(X))^2\} \leftarrow$ M.S.E.

Minimum MSE predictor is $E[Y/X]$

$E\{(Y-g(X))^2\} \geq E\{(Y-E[Y|X])^2\}$

Linear Prediction

Linear Prediction

$g(X) = a + bX$

$\mathcal{L}(a,b) = E\{(Y-a-bX)^2\}$

$\frac{\partial \mathcal{L}}{\partial a} = -2E[Y] + 2a + 2b\,E[X] = 0$

$\frac{\partial \mathcal{L}}{\partial b} = -2E[XY] + 2a\,E[X] + 2b\,E[X^2] = 0$

Solve for a, b:

$g(X) = \mu_Y + \rho \frac{\sigma_{XY}}{\sigma_X}(X - \mu_X)$

Limit Theorems

Limit Theorems

Suppose we have a sequence $X_1, X_2 \cdots$ of ~~iden~~ independent, identically distributed (i.i.d) rv's, what, if anything, can we say about

$Z = X_1 + X_2 + \cdots X_n$?

Limit Thms tell us about Z as $n \to \infty$, in practice, use results for large, but finite, n.

→ Average of X_i's [weak law of Large #'s]

→ distribution Z [central Limit Thm]

Require μ (finite), σ^2, independence of X_i

Bounds on Probabilities

Bounds on Probabilities

$P\{X > \alpha\} = \int_\alpha^\infty f_X(u)\,du = \int_{-\infty}^\infty g(u) f_X(u)\,du$

with $g(u) \begin{cases} 1 & : u \geq \alpha \\ 0 & : \text{else} \end{cases}$ | unit step.

If $h(u) \geq g(u)$ for all u, then

$\int g(u) f_X(u)\,du \leq \int h(u) f_X(u)\,du$

Markov Inequality

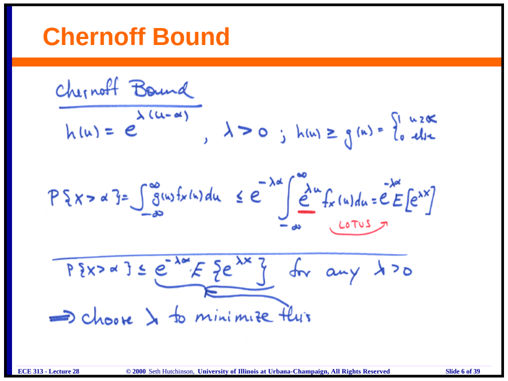
Markov Inequality

Let $X \geq 0$, μ finite.

$h(u) = \frac{u}{\alpha} \geq g(u) = \begin{cases} 1 & u \geq \alpha \\ 0 & \text{else} \end{cases}$

$P\{X > \alpha\} = \int_0^\infty g(u) f_X(u)\,du \leq \int_0^\infty \frac{u}{\alpha} f_X(u)\,du = \frac{\mu}{\alpha}$

For $\alpha < \mu \to$ Not Interesting


Chernoff Bound
Chernoff Bound
$h(u) = e^{\lambda(u-\alpha)}$, $\lambda > 0$; $h(u) \ge g(u) = \begin{cases} 1 & u \ge \alpha \\ 0 & \text{else} \end{cases}$
$P\{X > \alpha\} = \int_{-\infty}^{\infty} g(u) f_X(u)\,du \le e^{-\lambda\alpha} \int_{-\infty}^{\infty} e^{\lambda u} f_X(u)\,du = e^{-\lambda\alpha} E[e^{\lambda X}]$
LOTUS
$P\{X > \alpha\} \le e^{-\lambda\alpha} E\{e^{\lambda X}\}$ for any $\lambda > 0$
$\Rightarrow$ choose λ to minimize this
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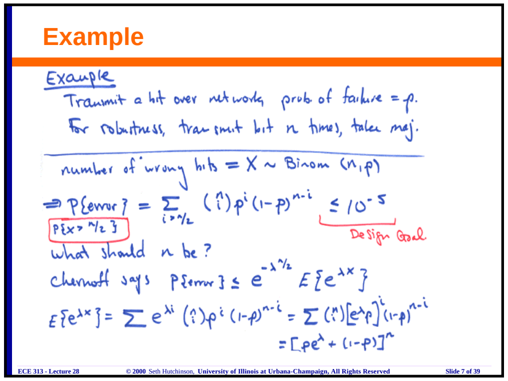
Example
Example
Transmit a bit over network prob. of failure = p.
For robustness, transmit bit n times, take maj.
number of wrong bits = $X \sim \text{Binom}(n, p)$
$\Rightarrow P\{\text{error}\} = \sum_{i > n/2} \binom{n}{i} p^i (1-p)^{n-i} \le 10^{-5}$
$P\{X > n/2\}$
Design Goal
what should n be?
Chernoff says $P\{\text{error}\} \le e^{-\lambda n/2} E\{e^{\lambda X}\}$
$E\{e^{\lambda X}\} = \sum e^{\lambda i} \binom{n}{i} p^i (1-p)^{n-i} = \sum \binom{n}{i} [e^{\lambda} p]^i (1-p)^{n-i}$
$= [pe^{\lambda} + (1-p)]^n$
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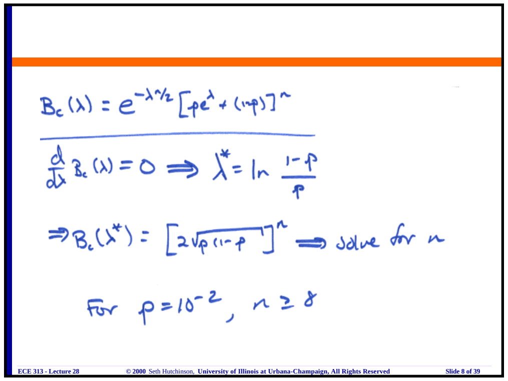
$B_c(\lambda) = e^{-\lambda n/2} [pe^{\lambda} + (1-p)]^n$
$\frac{d}{d\lambda} B_c(\lambda) = 0 \Rightarrow \lambda^* = \ln \frac{1-p}{p}$
$\Rightarrow B_c(\lambda^*) = [2\sqrt{p(1-p)}]^n \Rightarrow$ solve for n
For $p = 10^{-2}$, $n \ge 8$
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