ECE 313

Probability with Engineering Applications

Lecture 39 – Conditional Expectation

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Last Time

Last Time $f_{X|Y}(u|v) = \dfrac{f_{XY}(u,v)}{f_Y(v)}$ for $f_Y(v) > 0$

Computing Cond. Prob's:

$$P\{Y \in \underline{A}\} = \int_A f_Y(v)\,dv$$

condition on $X=u$:

$$P\{\underline{Y} \in A \mid X=u\} = \int_A f_{Y|X}(v|u)\,dv$$

relates to r.v. observed value

$$\cancel{F_{X|A}}\ F_{X|Y}(a|v) = \int_{-\infty}^{a} f_{X|Y}(u|v)\,du$$

Total Probability

Total Probability

$$f_Y(v) = \underbrace{\int_{-\infty}^{\infty} f_{XY}(u,v)\,du}_{\text{margin.}} = \int_{-\infty}^{\infty} \underbrace{f_{Y|X}(v|u)\, f_X(u)}_{f_{XY}(u,v)}\,du$$

$$P\{Y \in A\} = \int_A f_Y(v)\,dv = \int_A \left[\int_{-\infty}^{\infty} f_{Y|X}(v|u)\, f_X(u)\,du\right] dv$$

$$= \int_{-\infty}^{\infty} \underbrace{\left[\int_A f_{Y|X}(v|u)\,dv\right]} f_X(u)\,du$$

$$= \int_{-\infty}^{\infty} P\{Y \in A \mid X=u\}\ f_X(u)\,du$$

$$f_{Y|X}(v|u) = \frac{f_{XY}(u,v)}{f_X(v)} = \frac{f_{XY}(u,v)}{\int_{-\infty}^{\infty} f_{XY}(u,v)\,dv} \leftarrow \text{margin.}$$

$f_{Y|X}$ is a valid pdf

- $\int f_{Y|X}(v|u)\,dv = 1$
- $f_{Y|X}(v|u) \geq 0$ for all v

If X & Y independent

$$f_{Y|X}(v|u) = f_Y(v)$$

Conditional Expectation

Conditional Expectation:

Def: For X & Y discrete, the cond. expectation of X given that $Y=v$

$$E[X \mid Y=v] = \sum_i u_i\, p_{X|Y}(u_i|v)$$

if $p_Y(v) > 0$.

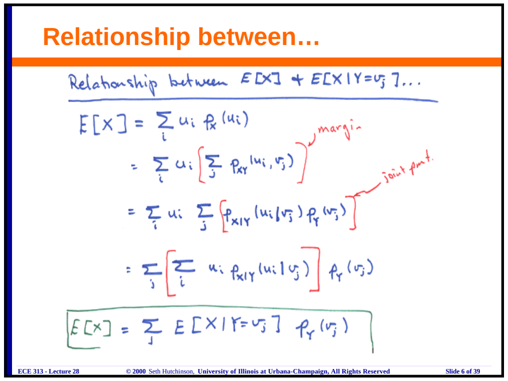
Def: X & Y ~~conts~~ jointly continuous:

$$E[X \mid Y=v] = \int_{-\infty}^{\infty} u\, f_{X|Y}(u|v)\,du$$

$f_Y(v) > 0$.

Conditional Expectations are…

- Conditional Expectations are expectations.
  - $E[g(X) \mid Y=v] = \begin{cases} \sum g(u_i)\, p_{X|Y}(u_i|v) \\ \int g(u)\, f_{X|Y}(u|v)\,dv \end{cases}$
  - $E[\sum X_i \mid Y=v] = \sum E[X_i \mid Y=v]$ (linearity)


Relationship between…
Relationship between $E[X]$ + $E[X|Y=v_j]$…
$E[X] = \sum_i u_i\, p_X(u_i)$
$= \sum_i u_i \left[\sum_j p_{XY}(u_i, v_j)\right]$
margin
$= \sum_i u_i \sum_j \left[p_{X|Y}(u_i|v_j)\, p_Y(v_j)\right]$
joint pmf.
$= \sum_j \left[\sum_i u_i\, p_{X|Y}(u_i|v_j)\right] p_Y(v_j)$
$\boxed{E[X] = \sum_j E[X|Y=v_j]\; p_Y(v_j)}$
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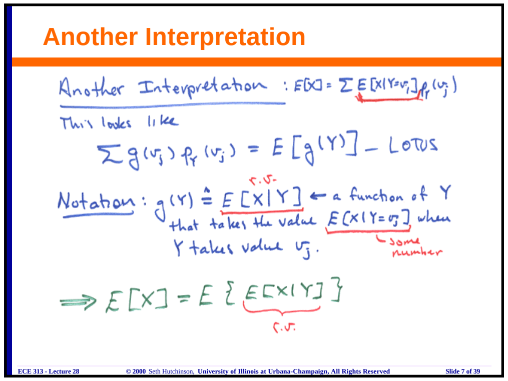
Another Interpretation
Another Interpretation : $E[X] = \sum E[X|Y=v_j]\, p_Y(v_j)$
This looks like
$\sum g(v_j)\, p_Y(v_j) = E[g(Y)]$ – LOTUS
Notation: $g(Y) \triangleq \underbrace{E[X|Y]}^{\text{r.v.}}$ ← a function of Y
that takes the value $E[X|Y=v_j]$ when
Y takes value v_j.
some number
$\Longrightarrow E[X] = E\{\underbrace{E[X|Y]}_{\text{r.v.}}\}$
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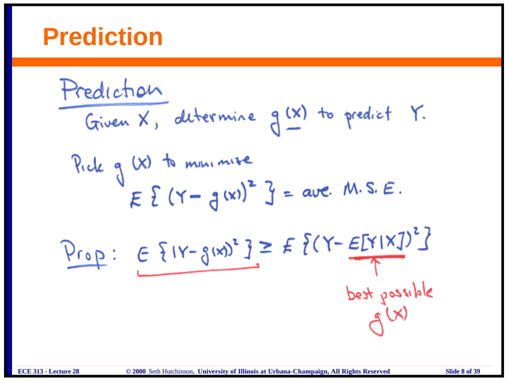
Prediction
Prediction
Given X, determine $g(X)$ to predict Y.
Pick $g(X)$ to minimize
$E\{(Y - g(X))^2\}$ = ave. M.S.E.
Prop: $E\{(Y-g(X))^2\} \geq E\{(Y - E[Y|X])^2\}$
best possible $g(X)$
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