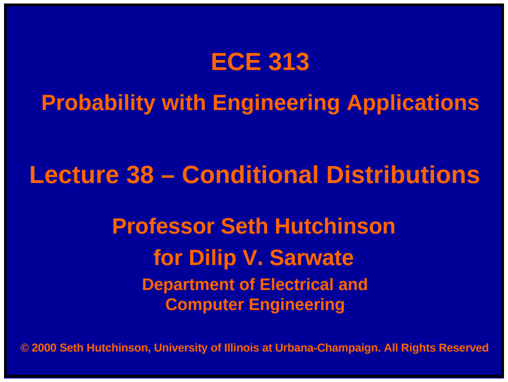
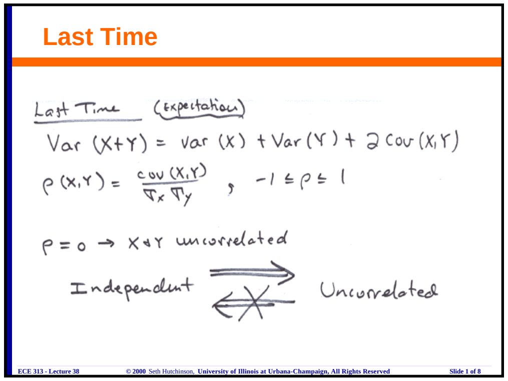
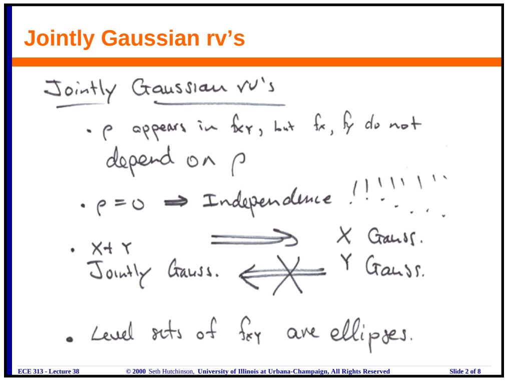
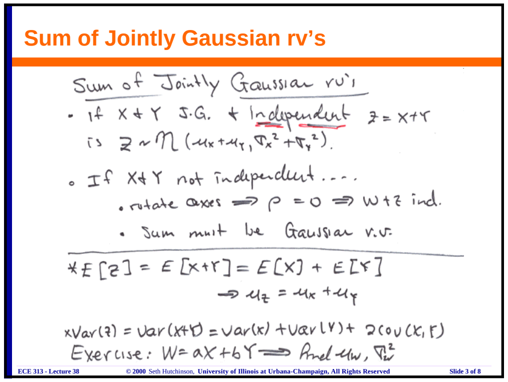
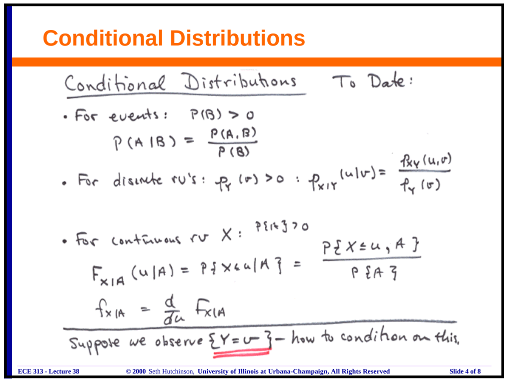
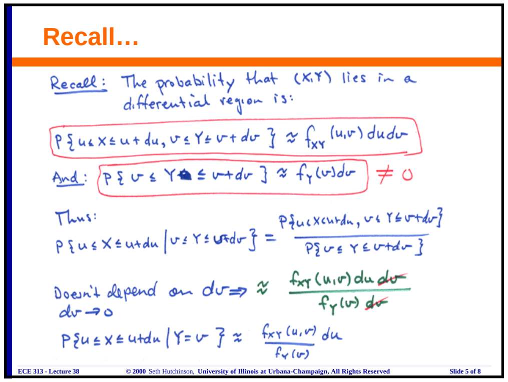

ECE 313
Probability with Engineering Applications
Lecture 38 – Conditional Distributions
Professor Seth Hutchinson
for Dilip V. Sarwate
Department of Electrical and
Computer Engineering
© 2000 Seth Hutchinson, University of Illinois at Urbana-Champaign. All Rights Reserved


Last Time
Last Time (expectation)
$Var(X+Y) = var(X) + Var(Y) + 2\,Cov(X,Y)$
$\rho(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y}$, $-1 \le \rho \le 1$
$\rho = 0 \rightarrow X \,\&\, Y$ uncorrelated
Independent $\Longrightarrow$ Uncorrelated
Uncorrelated $\not\Longrightarrow$ Independent
ECE 313 - Lecture 38 © 2000 Seth Hutchinson, University of Illinois at Urbana-Champaign, All Rights Reserved Slide 1 of 8


Jointly Gaussian rv's
Jointly Gaussian rv's
• ρ appears in f_{XY}, but f_X, f_Y do not depend on ρ
• $\rho = 0 \Rightarrow$ Independence !!!!!!
• X & Y Jointly Gauss. $\Longrightarrow$ X Gauss. Y Gauss.
• Level sets of f_{XY} are ellipses.
ECE 313 - Lecture 38 © 2000 Seth Hutchinson, University of Illinois at Urbana-Champaign, All Rights Reserved Slide 2 of 8


Sum of Jointly Gaussian rv's
Sum of Jointly Gaussian rv's
• If X & Y J.G. & Independent $Z = X+Y$ is $Z \sim \mathcal{N}(\mu_X+\mu_Y, \sigma_X^2+\sigma_Y^2)$.
• If X & Y not independent....
• rotate axes $\Rightarrow \rho = 0 \Rightarrow$ W & Z ind.
• Sum must be Gaussian r.v.
$*E[Z] = E[X+Y] = E[X] + E[Y]$
$\Rightarrow \mu_Z = \mu_X + \mu_Y$
$*Var(Z) = Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$
Exercise: $W = aX + bY \Longrightarrow$ Find μ_W, σ_W^2
ECE 313 - Lecture 38 © 2000 Seth Hutchinson, University of Illinois at Urbana-Champaign, All Rights Reserved Slide 3 of 8


Conditional Distributions
Conditional Distributions To Date:
• For events: $P(B) > 0$
$P(A|B) = \frac{P(A,B)}{P(B)}$
• For discrete rv's: $p_Y(v) > 0$: $p_{X|Y}(u|v) = \frac{p_{XY}(u,v)}{p_Y(v)}$
• For continuous rv X: $P\{A\} > 0$
$F_{X|A}(u|A) = P\{X \le u | A\} = \frac{P\{X \le u, A\}}{P\{A\}}$
$f_{X|A} = \frac{d}{du} F_{X|A}$
Suppose we observe $\{Y = v\}$ – how to condition on this.
ECE 313 - Lecture 38 © 2000 Seth Hutchinson, University of Illinois at Urbana-Champaign, All Rights Reserved Slide 4 of 8


Recall…
Recall: The probability that (X,Y) lies in a differential region is:
$P\{u \le X \le u+du, v \le Y \le v+dv\} \approx f_{XY}(u,v)\,du\,dv$
And: $P\{v \le Y \le v+dv\} \approx f_Y(v)\,dv$ $\neq 0$
Thus:
$P\{u \le X \le u+du \,|\, v \le Y \le v+dv\} = \frac{P\{u \le X \le u+du, v \le Y \le v+dv\}}{P\{v \le Y \le v+dv\}}$
Doesn't depend on $dv \Rightarrow \approx \frac{f_{XY}(u,v)\,du\,\cancel{dv}}{f_Y(v)\,\cancel{dv}}$
$dv \rightarrow 0$
$P\{u \le X \le u+du \,|\, Y = v\} \approx \frac{f_{XY}(u,v)}{f_Y(v)}\,du$
ECE 313 - Lecture 38 © 2000 Seth Hutchinson, University of Illinois at Urbana-Champaign, All Rights Reserved Slide 5 of 8

$$f_{X|Y}(u|v)\,du \approx \frac{f_{XY}(u,v)\,du}{f_Y(v)}$$

$\Rightarrow$ Define $f_{X|Y}(u|v) = \dfrac{f_{XY}(u,v)}{f_Y(v)}$

for $f_Y(v) \neq 0$

<u>Notation</u> $f_{X|Y}(u|v)$

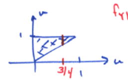
known

variable

$f_{X|Y}(u|3/4)$

Example

<u>Ex.</u> $f_{XY}(u,v) = \begin{cases} 2: & 0<u<v<1 \\ 0: & \text{else} \end{cases}$



$f_{Y|X}(v|3/4)$ $\Longrightarrow$

$f_{Y|X}(v|3/4)$

$\rightarrow f_X(u) = 2(1-u) \quad 0<u<1$

$\rightarrow f_{Y|X}(v|u) = \dfrac{2}{2(1-u)} = \underbrace{\dfrac{1}{1-u}}, \quad u<v<1$

uniform on $(u,1)$

Example

<u>EX</u> X + Y Joint. Gauss. $\mu_X = \mu_Y = 0$, $\sigma_X^2 = \sigma_Y^2 = 1$

$$f_{Y|X}(v|u) = \frac{f_{XY}(u,v)}{f_X(u)} = \frac{\frac{1}{2\pi\sqrt{1-\rho^2}} e^{-\frac{1}{2(1-\rho^2)}[u^2-2\rho uv+v^2]}}{\frac{1}{\sqrt{2\pi}} e^{-u^2/2}}$$

$$= \underbrace{\frac{1}{\sqrt{2\pi}\sqrt{1-\rho^2}} e^{-(v-\rho u)^2/2(1-\rho^2)}}_{}$$

$\mathcal{N}(\rho u, 1-\rho^2)$

