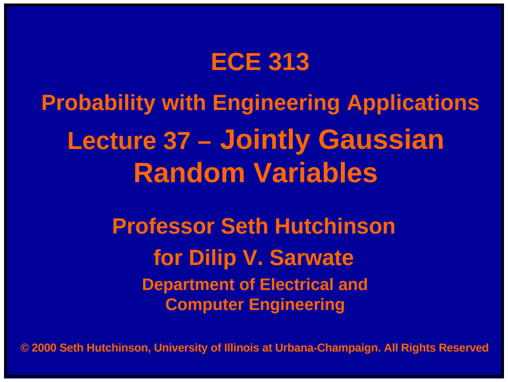
ECE 313
Probability with Engineering Applications
Lecture 37 – Jointly Gaussian Random Variables
Professor Seth Hutchinson
for Dilip V. Sarwate
Department of Electrical and Computer Engineering
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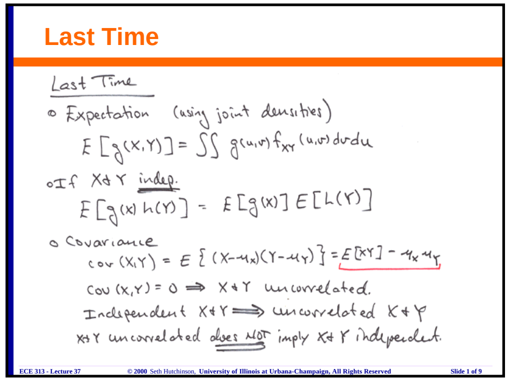
Last Time
Last Time
○ Expectation (using joint densities)
$E[g(X,Y)] = \iint g(u,v) f_{XY}(u,v)\, dv\, du$
○ If X & Y indep.
$E[g(X)h(Y)] = E[g(X)]\,E[h(Y)]$
○ Covariance
$cov(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\} = E[XY] - \mu_X\mu_Y$
$cov(X,Y) = 0 \Rightarrow$ X & Y uncorrelated.
Independent X & Y $\Longrightarrow$ uncorrelated X & Y
X & Y uncorrelated does NOT imply X & Y independent.
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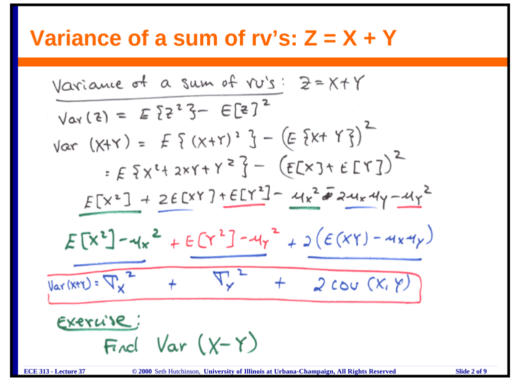
Slide 1 of 9


Variance of a sum of rv's: Z = X + Y
Variance of a sum of rv's: Z = X + Y
$Var(Z) = E\{Z^2\} - E[Z]^2$
$Var(X+Y) = E\{(X+Y)^2\} - (E\{X+Y\})^2$
$= E\{X^2 + 2XY + Y^2\} - (E[X] + E[Y])^2$
$E[X^2] + 2E[XY] + E[Y^2] - \mu_X^2 - 2\mu_X\mu_Y - \mu_Y^2$
$E[X^2] - \mu_X^2 + E[Y^2] - \mu_Y^2 + 2(E(XY) - \mu_X\mu_Y)$
$\boxed{Var(X+Y) = \sigma_X^2 + \sigma_Y^2 + 2\,cov(X,Y)}$
Exercise:
Find Var(X−Y)
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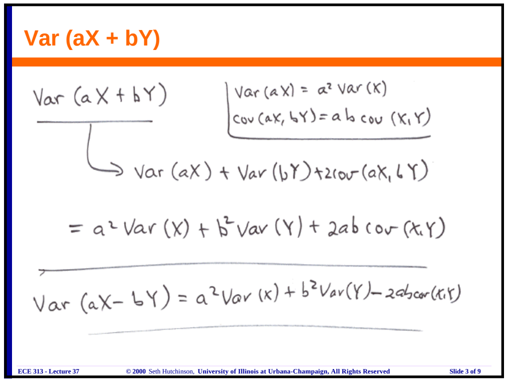
Var (aX + bY)
Var (aX + bY)
$Var(aX) = a^2 Var(X)$
$cov(aX, bY) = ab\, cov(X,Y)$
$\to Var(aX) + Var(bY) + 2cov(aX, bY)$
$= a^2 Var(X) + b^2 Var(Y) + 2ab\, cov(X,Y)$
$Var(aX - bY) = a^2 Var(X) + b^2 Var(Y) - 2ab\,cov(X,Y)$
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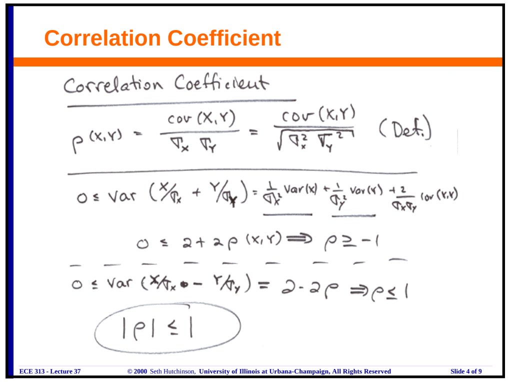
Slide 3 of 9


Correlation Coefficient
Correlation Coefficient
$\rho(X,Y) = \frac{cov(X,Y)}{\sigma_X \sigma_Y} = \frac{cov(X,Y)}{\sqrt{\sigma_X^2 \sigma_Y^2}}$ (Def.)
$0 \le Var(X/\sigma_X + Y/\sigma_Y) = \frac{1}{\sigma_X^2} Var(X) + \frac{1}{\sigma_Y^2} Var(Y) + \frac{2}{\sigma_X\sigma_Y} cov(X,Y)$
$0 \le 2 + 2\rho(X,Y) \Longrightarrow \rho \ge -1$
$0 \le Var(X/\sigma_X - Y/\sigma_Y) = 2 - 2\rho \Rightarrow \rho \le 1$
$|\rho| \le 1$
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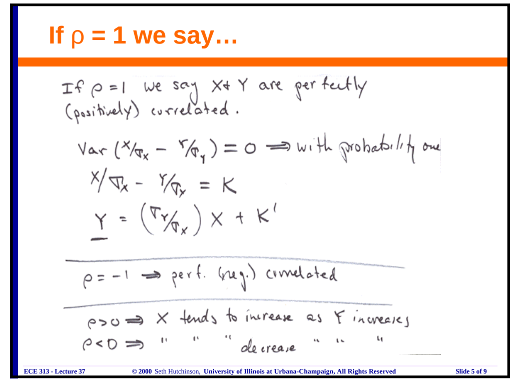
Slide 4 of 9


If ρ = 1 we say…
If $\rho = 1$ we say X & Y are perfectly (positively) correlated.
$Var(X/\sigma_X - Y/\sigma_Y) = 0 \Rightarrow$ with probability one
$X/\sigma_X - Y/\sigma_Y = K$
$Y = (\sigma_Y/\sigma_X)\, X + K'$
$\rho = -1 \Rightarrow$ perf. (neg.) correlated
$\rho > 0 \Rightarrow$ X tends to increase as Y increases
$\rho < 0 \Rightarrow$ " " " decrease " " "
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Slide 5 of 9

Jointly Gaussian rv's

Jointly Gaussian rv's

$X + Y$ are Joint. Gaussian if

$$f_{XY}(u,v) = C e^{-Q(u,v)}$$

$$C = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}}$$

$$Q(u,v) = \frac{1}{2(1-\rho^2)}\left[\frac{(u-\mu_X)^2}{\sigma_X^2} - 2\rho\underbrace{\frac{(u-\mu_X)(v-\mu_Y)}{\sigma_X\sigma_Y}} + \frac{(v-\mu_Y)^2}{\sigma_Y^2}\right]$$

$\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2$ are as usual

ρ is corr. coef. for $X + Y$, $\rho \neq 1$, $\rho \neq -1$

Marginal for Y:

Marginal for Y:

$$\underbrace{f_Y(v) = \frac{1}{\sqrt{2\pi}\,\sigma_Y} e^{-(v-\mu_Y)^2/2\sigma_Y^2}}$$

$\Rightarrow X + Y$ jointly Gaussian $\Rightarrow X$ Gauss. + Y Gauss.

$\Rightarrow X$ Gauss. + Y Gauss $\not\Rightarrow X + Y$ jointly Gauss.

Exercise:

$$f_{XY}(u,v) = \begin{cases} \frac{1}{\pi} e^{-(u^2+v^2)/2} & : \operatorname{sgn}(u) = \operatorname{sgn}(v) \\ 0 & : \text{else} \end{cases}$$

For ρ = 0 (i.e. X & Y uncorrelated)

For $\rho = 0$ (i.e. $X + Y$ uncorrelated)

$$Q(u,v) = \frac{1}{2}\left[\frac{(u-\mu_X)^2}{\sigma_X^2} + \frac{(v-\mu_Y)^2}{\sigma_Y^2}\right]$$

$$C = \frac{1}{\sqrt{2\pi}\,\sigma_X} \cdot \frac{1}{\sqrt{2\pi}\,\sigma_Y}$$

$$\Rightarrow C e^{-Q(u,v)} = \underbrace{\frac{1}{\sigma_X\sqrt{2\pi}} e^{-\frac{(u-\mu_X)^2}{2\sigma_X^2}}}_{f_X(u)} \cdot \underbrace{\frac{1}{\sigma_Y\sqrt{2\pi}} e^{-\frac{(v-\mu_Y)^2}{2\sigma_Y^2}}}_{f_Y(v)}$$

$\Longrightarrow X + Y$ independent!!!

How do they look?

How do they look

A contour of a surface defined by f_{XY} is the set

$$\{(u,v) \mid f_{XY}(u,v) = k\}$$

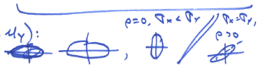
for some constant k. (aka level sets)

$$f_{XY}(u,v) = C e^{-Q(u,v)} = k \Rightarrow Q(u,v) = c$$

$$\underbrace{Q(u,v) = \frac{1}{2(1-\rho^2)}\left[\frac{(u-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{(u-\mu_X)(v-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(v-\mu_Y)^2}{\sigma_Y^2}\right]}$$

ellipse at (μ_X, μ_Y):

$\rho = 0$, $\sigma_X > \sigma_Y \Rightarrow$



$\rho = 0$, $\sigma_X < \sigma_Y$

$\sigma_X = \sigma_Y$, $\rho > 0$

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