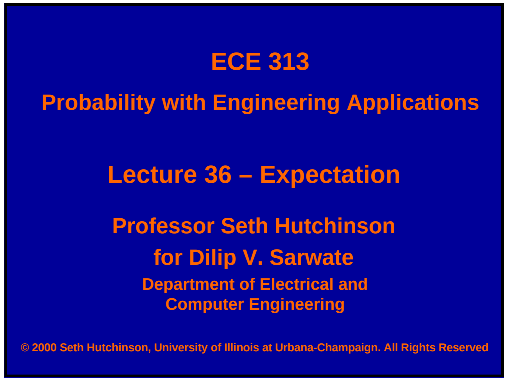
ECE 313
Probability with Engineering Applications
Lecture 36 – Expectation
Professor Seth Hutchinson
for Dilip V. Sarwate
Department of Electrical and Computer Engineering
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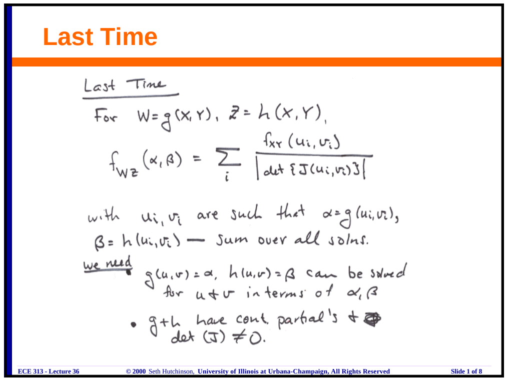
Last Time
Last Time
For W = g(X, Y), Z = h(X, Y),
f_WZ(α, β) = Σ_i f_XY(u_i, v_i) / |det{J(u_i, v_i)}|
with u_i, v_i are such that α = g(u_i, v_i), β = h(u_i, v_i) — sum over all solns.
we need
• g(u, v) = α, h(u, v) = β can be solved for u + v in terms of α, β
• g + h have cont. partial's + det(J) ≠ 0.
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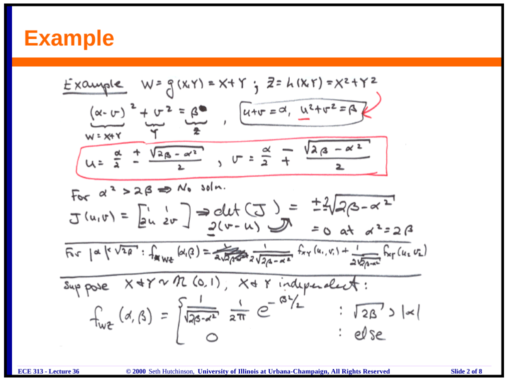
Example
Example W = g(X, Y) = X + Y ; Z = h(X, Y) = X² + Y²
(α − v)² + v² = β , u + v = α, u² + v² = β
W = X+Y Y Z
u = α/2 ± √(2β − α²)/2 , v = α/2 ∓ √(2β − α²)/2
For α² > 2β ⇒ No soln.
J(u, v) = [1 1; 2u 2v] ⇒ det(J) = 2(v − u) = ±2√(2β − α²)
= 0 at α² = 2β
For |α| < √(2β) : f_WZ(α, β) = 1/(2√(2β − α²)) f_XY(u₁, v₁) + 1/(2√(2β − α²)) f_XY(u₂, v₂)
Suppose X + Y ~ N(0, 1), X + Y independent:
f_WZ(α, β) = { 1/√(2β − α²) · 1/(2π) e^(−β/2) : √(2β) > |α| ; 0 : else
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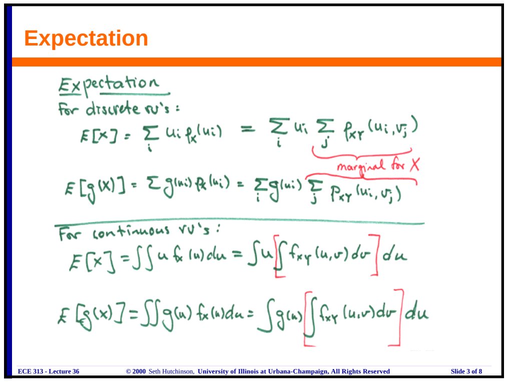
Expectation
Expectation
For discrete rv's:
E[X] = Σ_i u_i p_X(u_i) = Σ_i u_i Σ_j p_XY(u_i, v_j)
marginal for X
E[g(X)] = Σ g(u_i) p_X(u_i) = Σ_i g(u_i) Σ_j p_XY(u_i, v_j)
For continuous rv's:
E[X] = ∫∫ u f_X(u) du = ∫ u [∫ f_XY(u, v) dv] du
E[g(X)] = ∫∫ g(u) f_X(u) du = ∫ g(u) [∫ f_XY(u, v) dv] du
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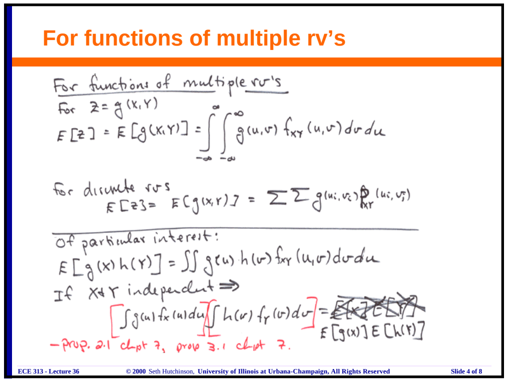
For functions of multiple rv's
For functions of multiple rv's
For Z = g(X, Y)
E[Z] = E[g(X, Y)] = ∫_{−∞}^{∞} ∫_{−∞}^{∞} g(u, v) f_XY(u, v) dv du
for discrete rv's
E[Z] = E[g(X, Y)] = Σ Σ g(u_i, v_j) p_XY(u_i, v_j)
Of particular interest:
E[g(X) h(Y)] = ∫∫ g(u) h(v) f_XY(u, v) dv du
If X + Y independent ⇒
[∫ g(u) f_X(u) du] [∫ h(v) f_Y(v) dv] = E[g(X)] E[h(Y)]
− prop. 2.1 chpt 7, prop 3.1 chpt 7.
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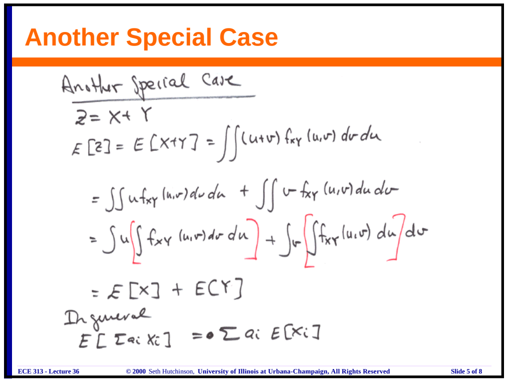
Another Special Case
Another Special Case
Z = X + Y
E[Z] = E[X + Y] = ∫∫ (u + v) f_XY(u, v) dv du
= ∫∫ u f_XY(u, v) dv du + ∫∫ v f_XY(u, v) du dv
= ∫ u [∫ f_XY(u, v) dv] du + ∫ v [∫ f_XY(u, v) du] dv
= E[X] + E[Y]
In general
E[Σ a_i X_i] = Σ a_i E[X_i]
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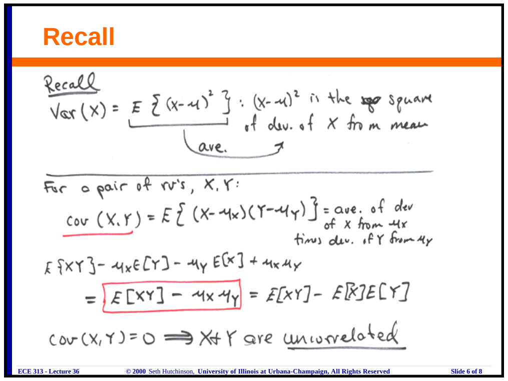
Recall
Recall
$Var(X) = E\{(X-\mu)^2\}$: $(X-\mu)^2$ is the square of dev. of X from mean
ave.
For a pair of rv's, X, Y:
$cov(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\}$ = ave. of dev of X from μ_X times dev. of Y from μ_Y
$E\{XY\} - \mu_X E[Y] - \mu_Y E[X] + \mu_X \mu_Y$
$= E[XY] - \mu_X \mu_Y = E[XY] - E[X]E[Y]$
$cov(X,Y) = 0 \Longrightarrow$ X & Y are uncorrelated
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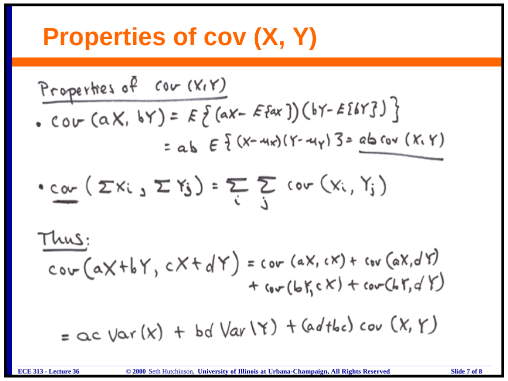
Properties of cov (X, Y)
Properties of cov (X,Y)
• $cov(aX, bY) = E\{(aX - E\{aX\})(bY - E\{bY\})\}$
$= ab\, E\{(X-\mu_X)(Y-\mu_Y)\} = ab\, cov(X,Y)$
• $cov(\Sigma X_i, \Sigma Y_j) = \sum_i \sum_j cov(X_i, Y_j)$
Thus:
$cov(aX+bY, cX+dY) = cov(aX, cX) + cov(aX, dY) + cov(bY, cX) + cov(bY, dY)$
$= ac\, Var(X) + bd\, Var(Y) + (ad+bc)\, cov(X,Y)$
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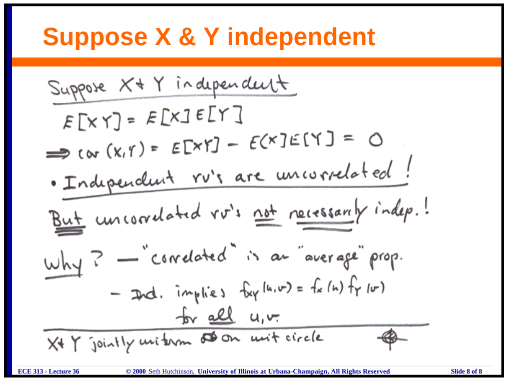
Suppose X & Y independent
Suppose X & Y independent
$E[XY] = E[X]E[Y]$
$\Longrightarrow cov(X,Y) = E[XY] - E[X]E[Y] = 0$
• Independent rv's are uncorrelated!
But uncorrelated rv's not necessarily indep.!
why? — "correlated" is an "average" prop.
– Ind. implies $f_{XY}(u,v) = f_X(u) f_Y(v)$ for all u, v
X & Y jointly uniform on unit circle
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