

**ECE 313**  
**Probability with Engineering Applications**  
**Lecture 35 – Multiple Functions of**  
**Multiple Random Variables**

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**Last Time**

Last Time ( $X + Y$  Independent)  
 $Z = X + Y \Rightarrow f_z(z) = \int_{-\infty}^{\infty} f_x(u) f_y(z-u) du$  \*

special case  
 $Y$  uniform on  $(-1, 1) \Rightarrow f_z(z) = \frac{1}{2} [F_x(z+1) - F_x(z-1)]$

$Z = aX + bY \Rightarrow f_z(z) = \frac{1}{|ab|} \int_{-\infty}^{\infty} f_x(\frac{z-u}{a}) f_y(\frac{u}{b}) du$  \*

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$Z$ 's of two r.v's  
 $Z = \min(X, Y), f_z(z) = [1 - F_x(z)] f_y(z) + f_x(z) [1 - F_y(z)]$

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**Hazard Rate fns**

Hazard Rate fns  $\lambda_z(t) = \frac{f_z(t)}{1 - F_z(t)}$ ;  $\lambda_z(t) dt$  = prob of failure w/in dt

$X, Y$  be lifetimes of two components,  $Z = \text{sys lifetime}$   
 $Z = \min(X, Y)$

we know:  $F_z(t) = 1 - (1 - F_x(t))(1 - F_y(t))$   
 $f_z(t) = f_x(t)[1 - F_y(t)] + f_y(t)[1 - F_x(t)]$

$\Rightarrow \frac{f_z(t)}{1 - F_z(t)} = \frac{f_x(t)}{1 - F_x(t)} + \frac{f_y(t)}{1 - F_y(t)}$   
 $\lambda_z(t) = \lambda_x(t) + \lambda_y(t)$

Exercise: Generalise this to  $Z = \min\{X_1, X_2, \dots, X_n\}$

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**Multiple fns of multiple rv's**

Multiple fns of multiple rv's  
 Given  $W = g(X, Y), Z = h(X, Y)$ ; given  $f_{XY}(u, v)$   
 Find:  $f_{WZ}(\alpha, \beta)$ .

General Approach:  
 $f_{WZ}(\alpha, \beta) = \iint f_{XY}(u, v) d u d v$   
 $\left. \begin{matrix} g(u, v) = \alpha \\ h(u, v) = \beta \end{matrix} \right\} \text{Too difficult}$

$\Rightarrow f_{WZ}(\alpha, \beta) = \frac{d^2}{d\alpha d\beta} F_{WZ}(\alpha, \beta)$

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$W = g(X, Y) = X + Y, Z = h(X, Y) = X - Y$

$P\{\alpha_1 \leq W \leq \alpha_1 + d\alpha, \beta_1 \leq Z \leq \beta_1 + d\beta\} = P\{\alpha_1 \leq X + Y \leq \alpha_1 + d\alpha, \beta_1 \leq X - Y \leq \beta_1 + d\beta\}$

$f_{WZ}(\alpha, \beta) d\alpha d\beta = \frac{1}{2} f_{XY}(\frac{\alpha + \beta}{2}, \frac{\alpha - \beta}{2}) d\alpha d\beta$

$u + v = \alpha_1 \Rightarrow u = \frac{\alpha_1 + d\alpha}{2}, v = \frac{\alpha_1 - d\alpha}{2}$   
 $u - v = \beta_1 \Rightarrow u = \frac{\alpha_1 + \beta_1 + d\alpha}{2}, v = \frac{\alpha_1 - \beta_1 - d\alpha}{2}$

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**Key Ideas**

Key Ideas — equate prob mass in differential area of  $u-v$  plane to prob mass in a differential area of  $w-z$  plane.

To do this, we need  
 ① relationship between  $A_{wz}, A_{uv}$   
 ② need to find  $u, v$  s.t.  $g(u, v) = \alpha, h(u, v) = \beta$

$A_{wz} = |\det [g(u, v), h(u, v)]| d u d v$

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$$A_{uv} = \left| \det [J(u_i, v_j)] \right|^{-1} \frac{d\alpha d\beta}{A_{\alpha\beta}}$$

where:  $g(u_i, v_j) = \alpha$ ,  $h(u_i, v_j) = \beta$

$$J(u_i, v_j) = \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix}$$

$$\Rightarrow f_{W_2}(\alpha, \beta) = \sum_i \frac{f_{XY}(u_i, v_i)}{|\det J(u_i, v_i)|}$$

for  $i$  ranging over all solutions  $g(u_i, v_i) = \alpha, h(u_i, v_i) = \beta$ .

$$g(x, y) = X + Y, \quad h(x, y) = X - Y$$

$$J = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \Rightarrow |\det [J]| = |-2| = 2$$

$$\underline{u} = \frac{\alpha + \beta}{2}, \quad \underline{v} = \frac{\alpha - \beta}{2}$$

$$f_{W_2}(\alpha, \beta) = \frac{1}{2} f_{XY}\left(\frac{\alpha + \beta}{2}, \frac{\alpha - \beta}{2}\right)$$

It works when

- we can solve  $g, h$  for  $u_i, v_i$
- $g+h$  have cont. partial derivatives  
 $\det [J] \neq 0$