ECE 313

Probability with Engineering Applications

Lecture 34 – Functions of Independent Random Variables

Professor Seth Hutchinson
for Dilip V. Sarwate
Department of Electrical and Computer Engineering

Last Time

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X & Y independent:

$$P\{X\in A, Y\in B\} = P\{X\in A\}P\{Y\in B\}$$

$$F_{XY}(u,v) = F_X(u)F_Y(v)$$

continuous rv's: $f_{XY}(u,v) = f_X(u)f_Y(v)$

discrete rv's: $p_{XY}(u,v) = p_X(u)p_Y(v)$

if. $f_{XY}(u,v) = \begin{cases} \frac{1}{(b-a)(d-c)} : & a<u<b,\ c<v<d \\ 0 : & \text{else} \end{cases}$

X & Y are ind, X is uniform on (a,b)

Y is uniform on (c,d)

Sums of Independent Discrete RV's

Sums of Ind. Discrete rv's

- X_i Binomial w/param (n_i,p), then $Z = \sum X_i$ is Binomial w/param (n,p), $n = \sum n_i$.
- If X_i are Poisson w/param λ_i, then $Z = \sum X_i$ is Poisson w/param $\lambda = \sum \lambda_i$.
- X_i Neg. Binom. w/params (n_i,p) then $Z = \sum X_i$ is neg. Binom. w/param (n,p) $n = \sum n_i$

Sums of Independent Continuous RV's

Sums of Ind. cont. rv's

We know: X & Y are jointly continuous, for $Z = X+Y$

$$f_Z(\alpha) = \int_{-\infty}^{\infty} f_{XY}(u,\alpha-u)\,du = \underbrace{\int_{-\infty}^{\infty} f_X(u)f_Y(\alpha-u)\,du}_{\text{convolution.}}$$

For the CDF

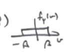
$$F_Z(\alpha) = P\{Z\le\alpha\} = P\{X+Y\le\alpha\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\alpha-u} f_X(u)f_Y(v)\,dv\,du = \int_{-\infty}^{\infty} F_Y(\alpha-u)f_X(u)\,du$$

$$= \int_{-\infty}^{\infty} F_X(\alpha-v)f_Y(v)\,dv$$

Example

Example $Z = X+Y$, Y uniform on $(-\beta,\beta)$



$$f_Z(\alpha) = \int_{-\infty}^{\infty} f_X(u)f_Y(\alpha-u)\,du$$

$$f_Y(\alpha-u) = \begin{cases}\frac{1}{2\beta} & -\beta<\alpha-u<\beta \\ & -\beta-\alpha<-u<\beta-\alpha \\ & \alpha-\beta<u<\alpha+\beta\end{cases}$$

$$f_Y(\alpha-u) = \begin{cases}\frac{1}{2\beta} & \alpha-\beta<u<\alpha+\beta \\ 0 & \text{else}\end{cases}$$

$$\Downarrow$$

$$\frac{1}{2\beta}\int_{\alpha-\beta}^{\alpha+\beta} f_X(u)\,du$$

$$\boxed{f_Z(\alpha) = \frac{1}{2\beta}\left[F_X(\alpha+\beta) - F_X(\alpha-\beta)\right]} = \frac{1}{2\beta}P\{\alpha-\beta<X<\alpha+\beta\}$$

Example

Example $Z = aX + bY$, $a,b>0$

$X' = aX$, $Y' = bY$

$$F_{X'}(u) = P\{aX\le u\} = P\{X\le u/a\} = F_X(u/a)$$

$$\Rightarrow \boxed{f_{X'}(u) = \frac{1}{|a|}f_X(u/a)} \leftarrow \text{for } a>0 \text{ or } a<0$$

$$f_{Y'}(v) = \frac{1}{b}f_Y(v/b)$$

$Z = X'+Y'$

$$\Rightarrow f_Z(\alpha) = \int_{-\infty}^{\infty} f_{X'}(v-\alpha)f_{Y'}(v)\,dv$$

$$\boxed{= \frac{1}{ab}\int_{-\infty}^{\infty} f_X\left(\frac{v-\alpha}{a}\right)f_Y\left(\frac{v}{b}\right)dv}$$

Special Cases

Special Cases

$X_i \sim \mathcal{N}(\mu_i, \sigma_i^2)$, X_i independent,

then $Z = \sum X_i$, $Z \sim \mathcal{N}(\mu, \sigma^2)$,

$\mu = \sum \mu_i$, $\sigma^2 = \sum \sigma_i^2$ (Ross pg. 268)

X + Y ind gamma rv's (s, λ) + (t, λ)

$Z = X + Y$ is gamma, $(s+t, \lambda)$

X_i ind. exponential w/param λ, then

$Z = \sum X_i$ is gamma w/param (n, λ)

(for n rv's X_i).

Other Functions of Continuous RV's

Other functions of continuous rv's

Basic approach: for $Z = g(X, Y)$ identify R, the region where $g(X,Y) \le \alpha$, then

$$F_Z(\alpha) = \iint_R f_{XY}(u,v)\,du\,dv = \iint_R f_X(u) f_Y(v)\,du\,dv$$

and
$$f_Z(\alpha) = \frac{d}{d\alpha} F_Z(\alpha)$$

Sometimes, we can find $F_Z(\alpha)$ without integration.

$Z = \max(X, Y)$ — X + Y ind.

$F_Z(\alpha) = P\{\max(X,Y) \le \alpha\} = P\{X \le \alpha, Y \le \alpha\}$

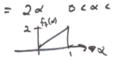
$= P\{X \le \alpha\} P\{Y \le \alpha\} = F_X(\alpha) F_Y(\alpha)$

$$f_Z(\alpha) = \frac{d}{d\alpha}\left[F_X(\alpha) F_Y(\alpha)\right] = f_X(\alpha) F_Y(\alpha) + F_X(\alpha) f_Y(\alpha)$$

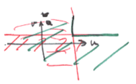
EX: X + Y uniform on (0,1)

$$F_X(\alpha) = F_Y(\alpha) = \begin{cases} \alpha & 0 < \alpha < 1 \\ 0 & \text{else} \end{cases}$$

$F_Z(\alpha) = \alpha^2 \quad 0 < \alpha < 1$, $f_Z(\alpha) = 2\alpha \quad 0 < \alpha < 1$



$Z = \cancel{\max}\ \min(X, Y)$



$F_Z(\alpha) = P\{\max(X,Y) \le \alpha\}$

$= P\left[\{X \le \alpha\} \cup \{Y \le \alpha\}\right]$ (**)

$= 1 - P\{X > \alpha, Y > \alpha\} = 1 - P\{X > \alpha\} P\{Y > \alpha\}$

$= 1 - (1 - F_X(\alpha))(1 - F_Y(\alpha))$ (*)

X + Y uniform on (0,1)

$F_Z(\alpha) = 1 - (1-\alpha)^2 \quad 0 < \alpha < 1$

$f_Z(\alpha) = 2(1-\alpha) \quad 0 < \alpha < 1$

(**) $F_X(\alpha) + F_Y(\alpha) - F_{XY}(\alpha, \alpha)$ (*)