

**ECE 313**  
**Probability with Engineering Applications**  
**Lecture 33 – Independent Random Variables**

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**Last Time**

Last Time  
 In general  $Z = g(X, Y)$ , to find  $f_Z(z)$   
 identify region of support,  $R$ , where  
 $g(X, Y) = z$   
 $f_Z(z) = \iint_R f_{X,Y}(x,y) dx dy$   
 Then  $f_Z(z) = f_X(z)$   
 → Trade out the integrals (esp limits)  
 For  $Z = X+Y$ :  $f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx$

**Example**

Example  
 $f_{X,Y}(x,y) = \begin{cases} 2e^{-2x-2y} & 0 < x, y < \infty \\ 0 & \text{else} \end{cases}$   
 $Z = Y/X \rightarrow -\infty < Z < \infty$   $f_Z(z) = ?$   
 $f_Z(z) = P\{Z=z\} = P\{Y/X=z\} = P\{Y=zX\}$   
 For  $z > 0$ :  $Y=zX$  when  $Y < \infty$   
 For  $z < 0$ :  $Y=zX$  when  $Y > 0$   
  
 $f_Z(z) = 2 \int_0^{\infty} \int_0^{\infty} e^{-2x-2y} dx dy = \left( \frac{1}{2} + \frac{\ln|z|}{2} \right)$

$f_Z(z) = \frac{d}{dz} \left( \frac{1}{2} + \frac{\ln|z|}{2} \right) = \frac{1}{2z} = \frac{1}{2|z|}$ ,  $-\infty < z < \infty$   
 → Cauchy random variable

**Independent Random Variables**

Independent Random Variables  
 $X$  &  $Y$  ind → knowing  $Y$  does not affect Prob of  $X$   
 $P\{X=A\}P\{Y=B\} = P\{X=A, Y=B\}$   
 $Z = X+Y \neq 0$   
 $P\{X=A, Y=B\} = \frac{P\{Z=A+B\}}{P\{Z=A+B\}}$   
 Not valid for ind!  
 →  $P\{X=A, Z=B\} = P\{X=A\}P\{Y=B\}$   
 $A = C - Y, B = Z - Y$   
 → For  $Z=B+Y$  → again ind!  
 $f_{X,Z}(x,z) = \frac{d}{dz} \int_{-\infty}^{\infty} f_{X,Y}(x,y) dy = \frac{d}{dz} \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dy = f_X(x) f_Y(z-x)$   
 Domain:  $x > 0, z > x$  for all  $x, y$

**Example**

Example  
 $f_{X,Y}(x,y) = \begin{cases} 2 & \text{otherwise} \\ 0 & \text{else} \end{cases}$   
  
 In general, a necessary (and sufficient) condition  
 for  $X$  &  $Y$  to be independent is that the joint pdf  
 is constant on an axis aligned rectangle [part/grid].  
  
 Maybe NO

### A Special Case

A special case: let  $X, Y$  be uniform on  $(a, b)$  and  $(c, d)$ .

for  $f(x, y) = \begin{cases} \frac{1}{(b-a)(d-c)} & ax+by \\ 0 & \text{else.} \end{cases}$

$g(x) = \int_{-\infty}^{\infty} f(x, y) dy = \int_{\frac{a-x}{b-a}}^{\frac{d-x}{b-a}} \frac{1}{(b-a)(d-c)} dy = \frac{1}{(b-a)(d-c)} (d-x-a)$

$\Rightarrow Y \sim U(a, b)$  correct

$\Rightarrow g(x) = 0 \Rightarrow X, Y$  ind

$X$  uniform on  $(a, b)$   
 $Y$  uniform on  $(c, d)$

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### Sums of Independent Discrete RVs

Sums of Independent Discrete Random Variables

let  $Z = X + Y$ ,  $X, Y$  take integer values

$P_Z(z) = P\{Z=z\} = P\{X+Y=z\} = \sum_{k=-\infty}^{\infty} P\{X=k, Y=z-k\}$

$\sum_{k=-\infty}^{\infty} P_X(k) P_Y(z-k)$  discrete convolution

Example  $X+Y$  Binomial distribution,  $X, Y$  Binomial

$Z = X+Y \sim \text{Bin}(n, p)$

$P_Z(z) = \sum_{k=0}^z \binom{n}{k} p^k (1-p)^{n-k} \binom{n}{z-k} p^{z-k} (1-p)^{n-(z-k)}$

$P_Z(z) = \binom{2n}{z} p^z (1-p)^{2n-z}$  Binomial distribution

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