

ECE 313
Probability with Engineering Applications
Lecture 32 – Functions of Multiple Random Variables

Professor Seth Hutchinson
for Dilip V. Sarwate
 Department of Electrical and
 Computer Engineering

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Last Time

For $X+Y$ jointly continuous


$$P\{(X,Y) \in B\} = \iint_B f_{XY}(u,v) du dv$$

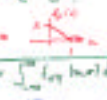
$$F_{XY}(u,v) = P\{X \leq u, Y \leq v\} = \int_{-\infty}^v \int_{-\infty}^u f_{XY}(u,v) du dv$$

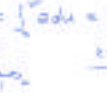
$$f_{XY}(u,v) = \frac{\partial^2}{\partial u \partial v} F_{XY}(u,v)$$

$$f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u,v) dv, \quad f_Y(v) = \int_{-\infty}^{\infty} f_{XY}(u,v) du$$


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$f_{XY}(u,v) = \begin{cases} 2 & 0 \leq u \leq v \leq 1 \\ 0 & \text{else} \end{cases}$


$f_X(u) = \begin{cases} 2(1-u) & 0 \leq u \leq 1 \\ 0 & \text{else} \end{cases}$


$f_Y(v) = \int_{-\infty}^v f_{XY}(u,v) du = \int_0^v 2 du = 2v$ for $0 \leq v \leq 1$


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$P\{1/2 \leq X \leq 2, 1/2 \leq Y \leq 2\}$


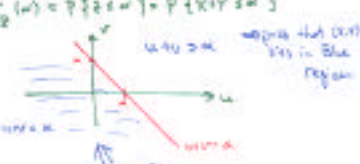
$$= \int_{1/2}^2 \int_{1/2}^u 2 dv du = \int_{1/2}^2 2(u-1/2) du = 1/4$$

$P\{0 \leq X \leq 1/2, 0 \leq Y \leq 1/2\} = 1/4$

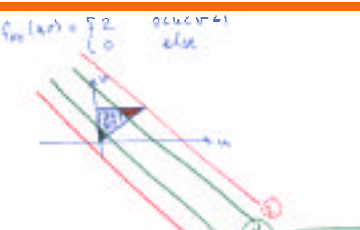
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Functions of Random Variables

Functions of Random Variables
 Input: cdf's $X+Y$ cdf's, $Z = X+Y$.
 → To find pdf for Z , find $F_Z(z)$, then differentiate

$$F_Z(z) = P\{Z \leq z\} = P\{X+Y \leq z\}$$


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$f_{XY}(u,v) = \begin{cases} 2 & 0 \leq u \leq v \leq 1 \\ 0 & \text{else} \end{cases}$


① $z < 0 \Rightarrow F_Z(z) = 0$
 ② $0 \leq z \leq 1 \Rightarrow F_Z(z) = 1/2 z^2$

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Case 3

Let $0 < \alpha < 1$

Area of triangle = $\frac{1}{2} \times \text{base} \times \text{height}$
 $\Rightarrow P_X(\alpha) = \frac{\alpha^2}{2} = \text{Area} \times \text{density} = \frac{\alpha^2}{2}$

Let $0 < \alpha < 1$
 $P_X(\alpha) = 1 - P_X(1-\alpha)$
 Area = $(1-\alpha)(1-\alpha) \times \frac{1}{2} = \frac{(1-\alpha)^2}{2}$
 $P_X(\alpha) = 1 - \frac{(1-\alpha)^2}{2}$

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$f_1(u) = \begin{cases} 0 & u < 0 \\ u/2 & 0 < u < 1 \\ 1 - (u-1)/2 & 1 < u < 2 \\ 0 & u > 2 \end{cases}$

$f_2(u) = \begin{cases} 0 & u < 0 \\ \alpha & 0 < u < 1 \\ 2-\alpha & 1 < u < 2 \\ 0 & u > 2 \end{cases}$

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More General Derivation

$f_Y(y) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{X_1}(u) f_{X_2}(v) \delta(y-u-v) du dv$

$f_Y(y) = \frac{d}{dy} F_Y(y) = \frac{d}{dy} \int_{-\infty}^y f_X(u) du = \int_{-\infty}^y f_X(u) \delta(y-u) du$

$f_Y(y) = \int_{-\infty}^{\infty} f_X(u) \delta(y-u) du$

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$f_{XY}(u,v) = \begin{cases} 2 & 0 < u < 1-u < 1 \\ 0 & \text{else} \end{cases}$

$0 < u < 1-u \Rightarrow 0 < u < \frac{1}{2}$

$1-u < u < 1 \Rightarrow u < 1-u \Rightarrow u < \frac{1}{2}$

$0 < u < \frac{1}{2} \Rightarrow \int_0^{1-u} 2 du = \alpha$

$1-u < u < 1 \Rightarrow \int_{u-1}^u 2 du = 2-\alpha$

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Another Example

Another example
 $f_{XY}(u,v) = \begin{cases} e^{-u^2-v^2} & 0 < u^2+v^2 < 1 \\ 0 & \text{else} \end{cases}$

$I = \iint e^{-u^2-v^2} du dv$

Let $r = \sqrt{u^2+v^2}$, $\theta = \tan^{-1}(v/u)$, $du dv = r dr d\theta$

$I = \int_0^{2\pi} \int_0^1 e^{-r^2} r dr d\theta = \frac{2\pi}{2} = \pi$

$\Rightarrow C = \frac{1}{\pi}$

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For $X+Y=R$

For $X+Y=R$
 $P\{R < \alpha\}$

$F_Y(\alpha) = \int_0^\alpha \int_0^{\alpha-r} \frac{2}{\pi} e^{-u^2-v^2} du dv = \alpha^2$

$f_Y(\alpha) = \frac{d}{d\alpha} F_Y(\alpha) = 2\alpha$

$f_Y(\alpha) = \begin{cases} 0 & \alpha < 0 \\ 2\alpha & 0 < \alpha < 1 \\ 0 & \alpha > 1 \end{cases}$

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