ECE 313

Probability with Engineering Applications

Lecture 31 – Jointly Continuous Random Variables

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Quick Review

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For jointly distributed rv's X & Y, the joint CDF is:

$$F_{XY}(u,v) = P\{X \le u, Y \le v\}$$

$F_{XY}(u,v)$ – completely specifies joint behavior of X & Y.

$F_X(u)$ & $F_{XY}(v)$ do NOT.

For Discrete rv's – joint pmf

$$p_{XY}(u,v) = P\{X = u, Y = v\}$$

$p_{XY}(u,v) \ge 0$

$\sum_i \sum_j p_{XY}(u_i,v_j) = 1$

Marginals:

$$F_X(a) = F_{XY}(a,\infty), \quad F_Y(b) = F_{XY}(\infty,b)$$

$$p_X(u) = \sum_j p_{XY}(u,v_j), \quad p_Y(v) = \sum_i p_{XY}(u_i,v)$$

Jointly Continuous RV's

Jointly Cont. rv's

For X & Y rv's, ~~the same~~ if (X,Y) can take any value in some region, then X & Y are jointly continuous.

$\boxed{EX}$ $X^2 + Y^2 < 4 \Rightarrow$

$0 < X < 1$, $0 < Y < 1 \Rightarrow$

As Before

$$F_{XY}(u,v) = P\{X \le u, Y \le v\}$$

Specify probability density instead of mass

As with individual cont. rv's, we specify probability density instead of mass,

- $f_{XY}(u,v)$ = density of prob. mass at pt (u,v)
- $f_{XY}(u,v)$ defines a surface in 3D, $f_{XY}(u,v) \ge 0$

To find prob mass, integrate ~~under~~ to find volume under this surface

$$P\{(X,Y) \in B\} = \iint_B f_{XY}(u,v)\, dv\, du$$

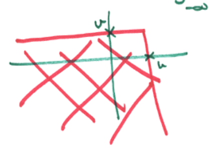
$$\Rightarrow \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(u,v)\, dv\, du = 1$$

Given $f_{xy}(u,v)$…

Given $f_{XY}(u,v)$, easy to find $F_{XY}(u,v)$:

$$F_{XY}(u,v) = P\{X \le u, Y \le v\} = \int_{-\infty}^{u}\int_{-\infty}^{v} f_{XY}(w,z)\, dz\, dw$$

$$= \int_{-\infty}^{v}\int_{-\infty}^{u} f_{XY}(w,z)\, dw\, dz$$



Given joint CDF, can we find joint pdf?

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First: how to differentiate an integral…

$$\underbrace{\frac{d}{ds}\left[\int_r^s g(u,v)\,du\right]}\; = \;?$$

Assume $G(u,v)$ is the antiderivative of $g(u,v)$ w.r.t. u

$$\frac{d}{ds}\left[G(s,v) - \underbrace{G(r,v)}_{\text{not a fn of } s}\right] = \frac{d}{ds} G(s,v)$$

$$= g(s,v)$$

$$\Rightarrow \boxed{\frac{d}{ds}\int_r^s g(u,v)\,du = g(s,v)}$$

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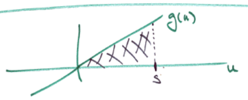
Example

<u>EXAMPLE</u> $g(u) = u$

$$\frac{d}{ds}\left[\int_0^s u\,du\right] = \frac{d}{ds}\left[\frac{u^2}{2}\Big|_0^s\right] = \frac{d}{ds}\,\frac{s^2}{2} = s$$

$$\underbrace{\frac{d}{ds}\int_0^s g(u)\,du = g(s)}$$ since $g(u)=u$, $g(s)=\underline{s}$



$$F_{XY}(a,b) = \int_{-\infty}^{a}\left(\int_{-\infty}^{b} f_{XY}(u,v)\,dv\right)du$$

Define $\int_{-\infty}^{b} f_{XY}(u,v)\,dv \triangleq g(u,b)$

$$F_{XY}(a,b) = \int_{-\infty}^{a} g(u,b)\,du$$

$$\frac{\partial}{\partial a}F_{XY}(a,b) = \underbrace{\frac{\partial}{\partial a}\int_{-\infty}^{a} g(u,b)\,du = g(a,b)}_{\text{Previous result.}}$$

$$g(a,b) = \int_{-\infty}^{b} f_{XY}(a,v)\,dv$$

$$\frac{\partial}{\partial b}\frac{\partial F_{XY}(a,b)}{\partial a} = \underbrace{\frac{\partial}{\partial b}\int_{-\infty}^{b} f_{XY}(a,v)\,dv}$$

$$\boxed{\frac{\partial^2}{\partial a\,\partial b}F_{XY}(a,b) = f_{XY}(a,b)}$$

The marginals for X & Y

<u>The marginals for X + Y:</u>

$$F_X(a) = F_{XY}(a,\infty) = P\{X \le a, Y \le \infty\} = \int_{-\infty}^{a}\int_{-\infty}^{\infty} f_{XY}(u,v)\,dv\,du$$

$$\frac{d}{da}F_X(a) = f_X(a) \Rightarrow f_X(a) = \frac{d}{da}\int_{-\infty}^{a}\int_{-\infty}^{\infty} f_{XY}(u,v)\,dv\,du$$

$$\boxed{f_X(a) = \underbrace{\int_{-\infty}^{\infty} f_{XY}(a,v)\,dv}_{\text{"integrate out" the Y component.}}}$$

$$\Rightarrow f_Y(b) = \int_{-\infty}^{\infty} f_{XY}(u,b)\,du$$

Uniform density

<u>Uniform density</u>

The random point (X,Y) is uniformly distributed on a region B if the joint pdf is

$$f_{XY}(u,v) = \begin{cases} c & (u,v)\in B \\ 0 & \text{else} \end{cases}$$

$$1 = \iint_B f_{XY}(u,v)\,dv\,du = c\iint_B dv\,du = c \times (\text{area of } B)$$

$$\Rightarrow c = \frac{1}{(\text{area of } B)}$$

Specific example

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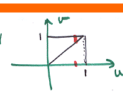
$$f_{XY}(u,v) = \begin{cases} 2 & 0<u<v<1 \\ 0 & \text{else} \end{cases}$$



$$f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u,v)\,dv$$

$f_{XY}(u,v) = 2$ for $0<u<v<1$, $f_{XY}(u,v) = 0$ else

$$\Rightarrow f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u,v)\,dv = \int_u^1 2\,dv = 2(1-u)$$

ex: $u = 3/4 \Rightarrow f_X(3/4) = \int_{3/4}^{1} 2\,dv = \frac{1}{2}$

$$f_X(u) = \begin{cases} 2(1-u) & 0<u<1 \\ 0 & \text{else} \end{cases}$$

Summary

<u>Summary</u>

$f_{XY}(u,v) \ge 0$ for all u, v

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(u,v)\,dv\,du = 1$$

$$F_{XY}(a,b) = P\{X \le a, Y \le b\} = \int_{-\infty}^{a}\int_{-\infty}^{b} f_{XY}(u,v)\,dv\,du$$

$$f_{XY}(a,b) = \frac{\partial^2}{\partial a\,\partial b}F_{XY}(a,b)$$

$$f_X(u) = \int_{-\infty}^{\infty} f_{XY}(u,v)\,dv$$

