

ECE 313
Probability with Engineering Applications
Lecture 30 – Jointly Distributed Random Variables

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Jointly Distributed Random Variables

Jointly Distributed Random Variables
 By now we know

- events $A, B \subseteq \mathcal{S}, P(A), \dots$
- single r.v.'s $X, f_X(u), P_X(u), F_X(u), \dots$
- functions of r.v.'s $Y = g(X), \dots, h(u)$
- Prob's for r.v.'s conditioned on event $P\{X=u | A\}$
 $f_{X|H_0}(u|H_0)$

EX: \mathcal{S} - Set of people
 exp. - select w at random
 $Y = g(X)$ $\left\{ \begin{array}{l} X = \text{height of } w \\ Y = \text{weight of } w \end{array} \right.$

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Given $P_X(u), P_Y(v)$ can I find $P_{XY}(u,v)$?

EX: $X \in \{0,1\}, Y \in \{0,1\}, P\{X=1\} = P\{Y=1\} = 1/2$

$P\{X=1 \cap Y=0\} = 1/4 \quad \left. \begin{array}{l} \phantom{P\{X=1 \cap Y=0\}} \\ \phantom{P\{X=1 \cap Y=0\}} \end{array} \right\} 1/6$
 $P\{X=1 \cap Y=1\} = 1/4 \quad \left. \begin{array}{l} \phantom{P\{X=1 \cap Y=0\}} \\ \phantom{P\{X=1 \cap Y=0\}} \end{array} \right\} 2/6$
 $P\{X=0 \cap Y=0\} = 1/4 \quad \left. \begin{array}{l} \phantom{P\{X=1 \cap Y=0\}} \\ \phantom{P\{X=1 \cap Y=0\}} \end{array} \right\} 2/6$
 $P\{X=0 \cap Y=1\} = 1/4 \quad \left. \begin{array}{l} \phantom{P\{X=1 \cap Y=0\}} \\ \phantom{P\{X=1 \cap Y=0\}} \end{array} \right\} 1/6$

$\{X=1\} = (\{X=1 \cap Y=0\} \cup \{X=1 \cap Y=1\})$

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Specify joint probability behavior...

To specify the joint prob. behavior, intro joint CDF.

Def: $F_{XY}(u,v) = P\{X \leq u \cap Y \leq v\}$
 $= P\{X \leq u, Y \leq v\}$ - Notation

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Special Cases

Special Cases

$F_{XY}(-\infty, b) = P\{X \leq -\infty, Y \leq b\} = 0$
 $F_{XY}(a, -\infty) = P\{X \leq a, Y \leq -\infty\} = 0$
 $F_{XY}(a, \infty) = P\{X \leq a, Y \leq \infty\} = P\{X \leq a\} = F_X(a)$
 $F_{XY}(\infty, b) = P\{X \leq \infty, Y \leq b\} = P\{Y \leq b\} = F_Y(b)$

Marginal CDFs

Property of CDF
 For a fixed a , $F_{XY}(a,v)$ is nondecreasing r.t. can h.o.r. v
 " " β , $F_{XY}(u,\beta)$ " " " " " " u

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Probabilities from joint CDFs

Probabilities from joint CDFs

$P\{a \leq X \leq b, c \leq Y \leq d\} = F_{XY}(b,d) - F_{XY}(a,d)$
 $- F_{XY}(b,c) + F_{XY}(a,c)$

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Discrete random variables

Discrete random variables
 $X \rightarrow Y$ s.t. $X \in \{u_1, u_2, \dots, u_n\}$
 $Y \in \{v_1, v_2, \dots, v_m\}$
 then (X, Y) takes its value from $\{u_1, u_2, \dots, u_n\} \times \{v_1, v_2, \dots, v_m\}$
 $\rightarrow (u_i, v_j)$ for $1 \leq i \leq n, 1 \leq j \leq m$
 The joint pmf for XY is
 $p_{XY}(u_i, v_j) = P\{X=u_i, Y=v_j\} = p_{ij}$

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$$\{X=u_i\} = \{X=u_i, Y=v_1\} \cup \{X=u_i, Y=v_2\} \cup \dots \cup \{X=u_i, Y=v_m\}$$

$$P\{X=u_i\} = P\{X=u_i, Y=v_1\} + \dots + P\{X=u_i, Y=v_m\}$$

$$= \sum_{j=1}^m P\{X=u_i, Y=v_j\}$$

$$p_X(u_i) = \sum_{j=1}^m p_{XY}(u_i, v_j) = \sum_{j=1}^m p_{ij}$$

marginal

$$p_Y(v_j) = \sum_{i=1}^n p_{XY}(u_i, v_j) = \sum_{i=1}^n p_{ij}$$

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$$\sum_{i=1}^n \sum_{j=1}^m p_{XY}(u_i, v_j) = 1$$

joint pmf
 $\sum_{i,j} p_{ij} = 1$

$$\sum_{i=1}^n p_X(u_i) = 1$$

$p_{XY}(u_i, v_j) \geq 0$ for all u_i, v_j

p_{11}	p_{12}	...	p_{1n}
\vdots	\vdots	\ddots	\vdots
p_{n1}	p_{n2}	...	p_{nm}

$\Rightarrow \sum = p_Y(v_j)$

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Conditional pmfs

The cond. pmf of X given that $Y=v_j$ is
 $p_{X|Y}(u_i | v_j) = \frac{P\{X=u_i, Y=v_j\}}{P\{Y=v_j\}} = \frac{p_{ij}}{p_Y(v_j)}$
 $= \frac{p_{XY}(u_i, v_j)}{p_Y(v_j)}$

$$\sum_{i=1}^n p_{X|Y}(u_i | v_j) = \sum_{i=1}^n \frac{p_{XY}(u_i, v_j)}{p_Y(v_j)} = \frac{1}{p_Y(v_j)} \sum_{i=1}^n p_{XY}(u_i, v_j) = \frac{p_Y(v_j)}{p_Y(v_j)} = 1$$

$p_{X|Y}(u_i | v_j) \geq 0$

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Bayes' Theorem

Bayes' Theorem:
 $p_{Y|X}(v_j | u_i) = \frac{p_{XY}(u_i, v_j) p_Y(v_j)}{p_X(u_i)}$

Independence:
 $p_{X|Y}(u_i | v_j) = p_X(u_i)$
 $p_{Y|X}(v_j | u_i) = \frac{p_{XY}(u_i, v_j)}{p_Y(v_j)} = p_X(u_i) \Rightarrow p_{XY}(u_i, v_j) = p_X(u_i) p_Y(v_j)$

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