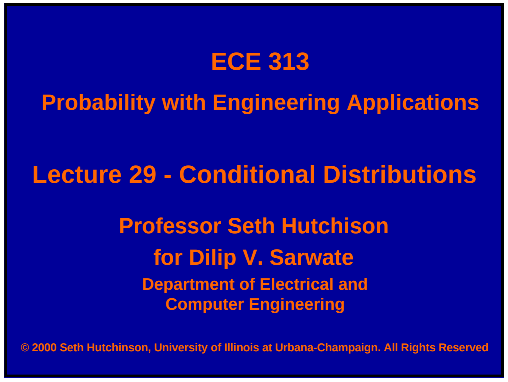
ECE 313
Probability with Engineering Applications
Lecture 29 - Conditional Distributions
Professor Seth Hutchinson
for Dilip V. Sarwate
Department of Electrical and
Computer Engineering
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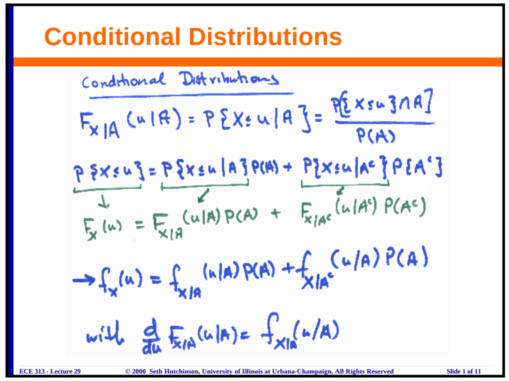
Conditional Distributions
Conditional Distributions
F_{X|A}(u|A) = P{X ≤ u | A} = P[{X ≤ u} ∩ A] / P(A)
P{X ≤ u} = P{X ≤ u | A} P(A) + P{X ≤ u | A^c} P{A^c}
F_X(u) = F_{X|A}(u|A) P(A) + F_{X|A^c}(u|A^c) P(A^c)
→ f_X(u) = f_{X|A}(u|A) P(A) + f_{X|A^c}(u|A) P(A)
with d/du F_{X|A}(u|A) = f_{X|A}(u/A)
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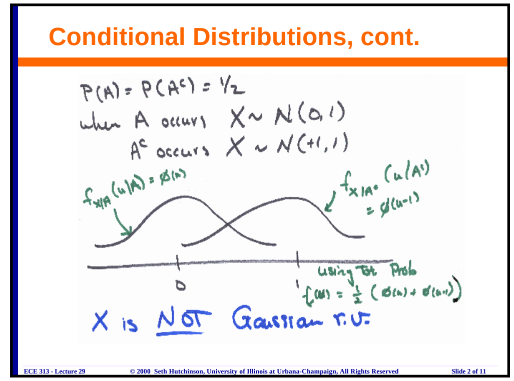
Conditional Distributions, cont.
P(A) = P(A^c) = 1/2
when A occurs X ~ N(0,1)
A^c occurs X ~ N(+1,1)
f_{X|A}(u|A) = φ(u)
f_{X|A^c}(u|A^c) = φ(u-1)
0
1
using tot. Prob
f(u) = 1/2 (φ(u) + φ(u-1))
X is NOT Gaussian r.v.
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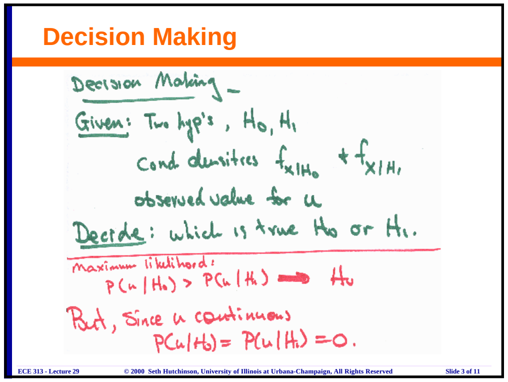
Decision Making
Decision Making –
Given: Two hyp's, H0, H1
Cond densities f_{X|H0} + f_{X|H1}
observed value for u
Decide: which is true H0 or H1.
maximum likelihood:
P(u | H0) > P(u | H1) ⟹ H0
But, since u continuous
P(u|H0) = P(u|H1) = 0.
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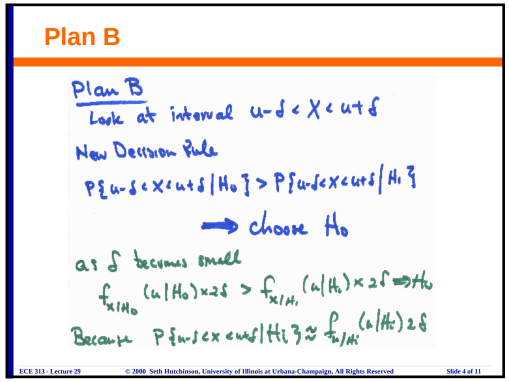
Plan B
Plan B
Look at interval u-δ < X < u+δ
New Decision Rule
P{u-δ < X < u+δ | H0} > P{u-δ < X < u+δ | H1}
⟹ choose H0
as δ becomes small
f_{X|H0}(u|H0)×2δ > f_{X|H1}(u|H1)×2δ ⟹ H0
Because P{u-δ < X < u+δ | Hi} ≈ f_{X|Hi}(u|Hi)2δ
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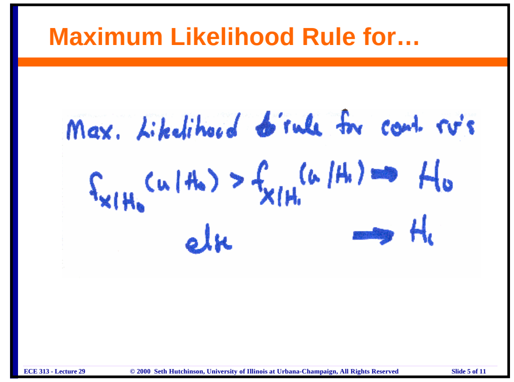
Maximum Likelihood Rule for…
Max. likelihood d. rule for cont. rv's
f_{X|H0}(u|H0) > f_{X|H1}(u|H1) ⟹ H0
else ⟹ H1
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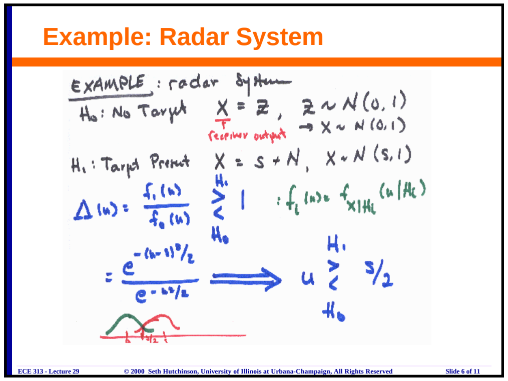
Example: Radar System
EXAMPLE : radar system
H0 : No Target X = Z, Z ~ N(0,1)
receiver output
→ X ~ N(0,1)
H1 : Target Present X = s + N, X ~ N(s,1)
Δ(u) = f1(u) / f0(u) ≷ 1
= e^{-(u-s)²/2} / e^{-u²/2} ⟹ u ≷ s/2
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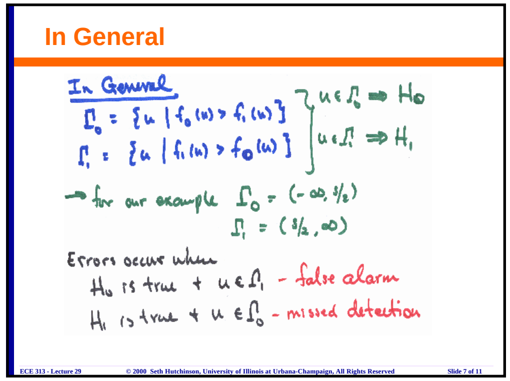
In General
In General
Γ0 = {u | f0(u) > f1(u)}
Γ1 = {u | f1(u) > f0(u)}
u ∈ Γ0 ⇒ H0
u ∈ Γ1 ⇒ H1
→ for our example Γ0 = (-∞, s/2)
Γ1 = (s/2, ∞)
Errors occur when
H0 is true + u ∈ Γ1 - false alarm
H1 is true + u ∈ Γ0 - missed detection
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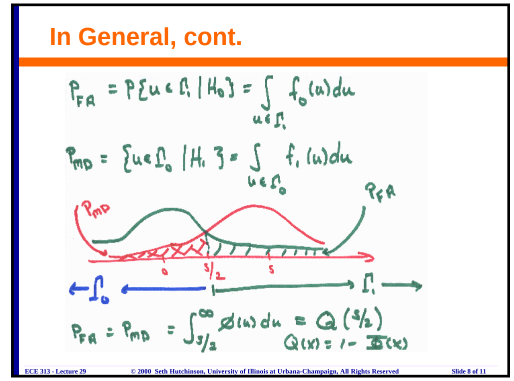
In General, cont.
P_FA = P{u ∈ Γ1 | H0} = ∫_{u∈Γ1} f0(u) du
P_MD = {u ∈ Γ0 | H1} = ∫_{u∈Γ0} f1(u) du
P_MD
P_FA
P_FA = P_MD = ∫_{s/2}^∞ φ(u) du = Q(s/2)
Q(x) = 1 - Φ(x)
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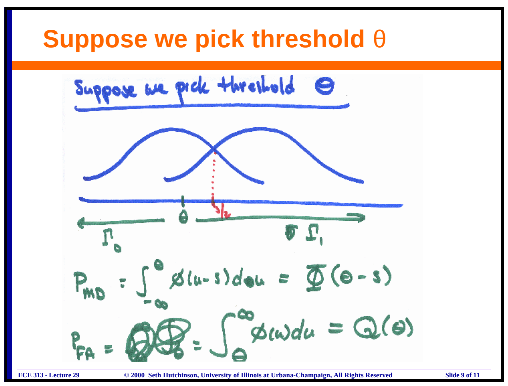
Suppose we pick threshold θ
Suppose we pick threshold θ
P_MD = ∫_{-∞}^θ φ(u-s) du = Φ(θ - s)
P_FA = ∫_θ^∞ φ(u) du = Q(θ)
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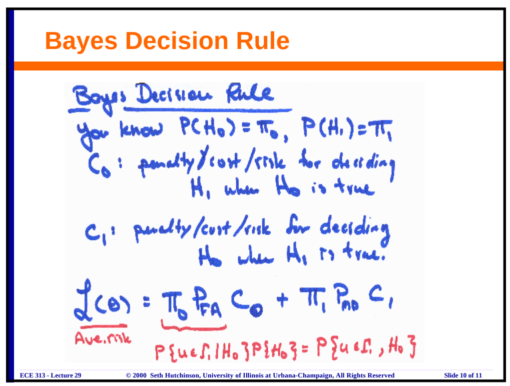
Bayes Decision Rule
Bayes Decision Rule
You know P(H0) = π0, P(H1) = π1
C0 : penalty/cost/risk for deciding H1 when H0 is true
C1 : penalty/cost/risk for deciding H0 when H1 is true.
ℒ(θ) = π0 P_FA C0 + π1 P_MD C1
Ave. risk
P{u ∈ Γ1 | H0} P{H0} = P{u ∈ Γ1, H0}
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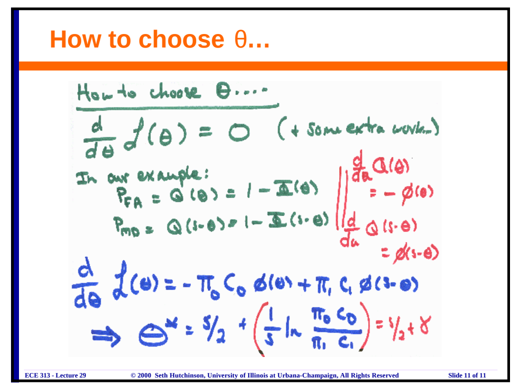
How to choose θ…
How to choose θ....
d/dθ ℒ(θ) = 0 (+ some extra work)
In our example:
P_FA = Q(θ) = 1 - Φ(θ)
P_MD = Q(s-θ) = 1 - Φ(s-θ)
d/du Q(θ) = -φ(θ)
d/du Q(s-θ) = φ(s-θ)
d/dθ ℒ(θ) = -π0 C0 φ(θ) + π1 C1 φ(s-θ)
⇒ θ* = s/2 + (1/s ln π0C0/π1C1) = s/2 + γ
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