Independent unit Gaussian RVs

- Let **W** and **Z** denote independent unit Gaussian random variables
- $f_W(\alpha) = (1/2\pi)^{1/2} \bullet \exp(-\alpha^2/2) = \varphi(\alpha)$
- $f_Z(\beta) = (1/2\pi)^{1/2} \bullet \exp(-\beta^2/2) = \varphi(\beta)$
  where $\varphi(\bullet)$ is the unit Gaussian pdf
- $f_{W,Z}(\alpha, \beta) = f_W(\alpha) \bullet f_Z(\beta) = \varphi(\alpha) \bullet \varphi(\beta)$
  $= (1/2\pi) \bullet \exp[-(\alpha^2 + \beta^2)/2]$
- This is a circularly symmetric pdf

Independent Gaussian RVs

- **W** and **Z** independent **N**(0, 1) RVs
- Let $\mathbf{X} = \sigma_X \bullet \mathbf{W} + \mu_X$ $\quad \mathbf{Y} = \sigma_Y \bullet \mathbf{Z} + \mu_Y$
- **X** and **Y** are $\mathbf{N}(\mu_X, (\sigma_X)^2)$ and $\mathbf{N}(\mu_Y, (\sigma_Y)^2)$ random variables respectively because linear functions of Gaussian RVs are Gaussian RVs
- **X** and **Y** are independent RVs because they are functions of independent RVs

pdf of independent Gaussian RVs

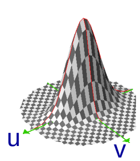
- $f_X(u) = (1/\sigma_X) \bullet \varphi((u-\mu_X)/\sigma_X)$
- $f_Y(v) = (1/\sigma_Y) \bullet \varphi((v-\mu_Y)/\sigma_Y)$
- $f_{X,Y}(u,v) = f_X(u) \bullet f_Y(v)$
  $= (1/\sigma_X) \bullet \varphi((u-\mu_X)/\sigma_X) \bullet (1/\sigma_Y) \bullet \varphi((v-\mu_Y)/\sigma_Y)$
  $= C \bullet \exp\{-([(u-\mu_X)/\sigma_X]^2 + [(v-\mu_Y)/\sigma_Y]^2)/2\}$
  where $C = 1/(2\pi\sigma_X\sigma_Y)$
- This is a not a circularly symmetric pdf
- The curve $f_{X,Y}(u,v) = c$ is an ellipse

The contours are ellipses

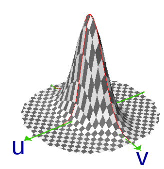
- **X** and **Y** are independent $\mathbf{N}(\mu_X, (\sigma_X)^2)$ and $\mathbf{N}(\mu_Y, (\sigma_Y)^2)$ RVs
- $f_{X,Y}(u,v) = f_X(u) \bullet f_Y(v)$
  $= C \bullet \exp\{-([(u-\mu_X)/\sigma_X]^2 + [(v-\mu_Y)/\sigma_Y]^2)/2\}$
  where $C = (1/2\pi\sigma_X\sigma_Y)$
- The curve $f_{X,Y}(u,v) = c$ defines an ellipse that is centered at (μ_X, μ_Y) and whose axes are parallel to the coordinate axes

2 pictures = 2000 words: Part I

- **X**, **Y** independent zero-mean Gaussians with variances $(\sigma_X)^2$ and $(\sigma_Y)^2$



$(\sigma_X)^2 > (\sigma_Y)^2$



$(\sigma_X)^2 < (\sigma_Y)^2$

Jointly Gaussian random variables

- The random variables in the previous few slides are simple examples of what are called jointly Gaussian random variables
- The RVs are independent and their marginal pdfs are Gaussian pdfs
- Generally, if **W** and **Z** are independent **N**(0, 1) RVs, then $\mathbf{X} = a\mathbf{W} + b\mathbf{Z} + \mu_X$ and $\mathbf{Y} = c\mathbf{W} + d\mathbf{Z} + \mu_Y$ are called jointly Gaussian random variables

Linear transformations

- If **W** and **Z** are independent **N**(0, 1) RVs, then $\mathbf{X} = a\mathbf{W} + b\mathbf{Z} + \mu_X$, $\mathbf{Y} = c\mathbf{W} + d\mathbf{Z} + \mu_Y$ are called jointly Gaussian RVs
- We previously considered the case $a = \sigma_X$, $b = 0$, $c = 0$, $d = \sigma_Y$, in which special case **X** and **Y** are independent
- In general, $\mathbf{X} = a\mathbf{W} + b\mathbf{Z} + \mu_X$ and $\mathbf{Y} = c\mathbf{W} + d\mathbf{Z} + \mu_Y$ are dependent Gaussian random variables

Means and variances

- **W** and **Z** are independent **N**(0, 1) RVs
- $E[\mathbf{X}] = E[a\mathbf{W} + b\mathbf{Z} + \mu_X] = \mu_X$
- $E[\mathbf{Y}] = E[c\mathbf{W} + d\mathbf{Z} + \mu_Y] = \mu_Y$
- Since $\text{cov}(\mathbf{W}, \mathbf{Z}) = 0$ (why?)
  $\text{var}(\mathbf{X}) = \text{var}(a\mathbf{W} + b\mathbf{Z} + \mu_X)$
  $= \text{var}(a\mathbf{W} + b\mathbf{Z}) = a^2 \bullet \text{var}(\mathbf{W}) + b^2 \bullet \text{var}(\mathbf{Z})$
  $= a^2 + b^2$
- Similarly, $\text{var}(\mathbf{Y}) = c^2 + d^2$

Covariance and correlation

- **W** and **Z** are independent **N**(0, 1) RVs
- $\text{cov}(\mathbf{X}, \mathbf{Y})$
  $= \text{cov}(a\mathbf{W} + b\mathbf{Z} + \mu_X, c\mathbf{W} + d\mathbf{Z} + \mu_Y)$
  $= ac \bullet \text{var}(\mathbf{W}) + bd \bullet \text{var}(\mathbf{Z}) + (ad+bc) \bullet \text{cov}(\mathbf{W},\mathbf{Z})$
  $= ac + bd$
- $\rho_{X,Y} = (ac+bd)/[(a^2 + b^2)(c^2 + d^2)]^{1/2}$
- $\rho_{X,Y}$ is the dot-product of the vectors [a,b] and [c,d] divided by their norms (lengths)

Once upon a matrix

- The covariance matrix of (**W**, **Z**) is the 2×2 identity matrix I
- $(\mathbf{X}, \mathbf{Y}) = (\mathbf{W}, \mathbf{Z})G + (\mu_X, \mu_Y)$ where G is the 2×2 matrix shown below
- $G = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$
- Covariance matrix of (**X**, **Y**) is $G^T I G = G^T G$
- $G^T G = \begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix}$

Joint pdf of X and Y – Approach

- Example T of Lecture 37: if $\underline{\mathbf{Y}} = \underline{\mathbf{X}}G$ where G is an n×n nonsingular matrix, then
  $f_{\underline{Y}}(\underline{v}) = f_{\underline{X}}(\underline{v}G^{-1})/|\det(G)|$
- $(\mathbf{X}-\mu_X, \mathbf{Y}-\mu_Y) = (\mathbf{W}, \mathbf{Z})G$
- If G is nonsingular, then the joint pdf of $(\mathbf{X}-\mu_X, \mathbf{Y}-\mu_Y)$ can be obtained
- $G = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $\quad G^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

Joint pdf of X and Y – u, v only

- $G = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ $\quad G^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$
- $f_{X-\mu_X, Y-\mu_Y}(u,v) = f_{W,Z}([u,v]G^{-1})/|\det(G)|$ where $\det(G) = ad - bc$
- Thus, we have expressed everything in terms of u and v
- But, plugging and chugging gives a messy formula that doesn't show the important stuff clearly

Joint pdf of X and Y — Re-write!

- var(**X**) = $a^2 + b^2$ var(**Y**) = $c^2 + d^2$
- cov((**X**,**Y**) = ac + bd det(G) = ad – bc
- $(ad - bc)^2 = (a^2 + b^2)(c^2 + d^2) - (ac + bd)^2$
- $[\det(G)]^2$ = var(**X**)•var(**Y**) – $[\text{cov}(\mathbf{X},\mathbf{Y})]^2$
- $f_{X-\mu_X, Y-\mu_Y}(u,v) = f_{W,Z}([u,v]G^{-1})/\det(G)$
- But $f_{W,Z}(\alpha, \beta) = (1/2\pi)\cdot\exp[-(\alpha^2 + \beta^2)/2]$
- So we need to set
  $\alpha = (du-bv)/\det(G)$ $\beta = (-cu+av)/\det(G)$

Joint pdf of X and Y – More algebra

- $\alpha^2 + \beta^2 = [(du-bv)^2 + (-cu+av)^2]/[\det(G)]^2$
- The numerator of this expression is
  u^2•var(**Y**) + v^2•var(**X**) –2(ac+bd)•uv
  = u^2•var(**Y**) + v^2•var(**X**) –2•cov(**X**,**Y**)•uv
- The denominator of the expression is
  $[\det(G)]^2$ = var(**X**)•var(**Y**) – $[\text{cov}(\mathbf{X},\mathbf{Y})]^2$
- Divide numerator and denominator by var(**X**)•var(**Y**)

Joint pdf of X and Y – Nearly done

- $\alpha^2 + \beta^2 = [(du-bv)^2 + (-cu+av)^2]/[\det(G)]^2$
  $= [(u/\sigma_X)^2 - 2\rho\cdot(u/\sigma_X)\cdot(v/\sigma_Y) + (v/\sigma_Y)^2]/(1-\rho^2)$
- $f_{X-\mu_X, Y-\mu_Y}(u,v) = f_{W,Z}([u,v]G^{-1})/\det(G)$
  = C•exp(–Q(u,v)) where
  $C = 1/2\pi\cdot\det(G) = 1/[2\pi\cdot\sigma_X\cdot\sigma_Y\cdot(1-\rho^2)^{1/2}]$
  and Q(u,v) is given by
  $[(u/\sigma_X)^2 - 2\rho\cdot(u/\sigma_X)\cdot(v/\sigma_Y) + (v/\sigma_Y)^2]/2(1-\rho^2)$

Joint pdf of X and Y – Final result

- $f_{X-\mu_X, Y-\mu_Y}(u,v) = f_{W,Z}([u,v]G^{-1})/\det(G)$
  = C•exp(–Q(u,v)) where
  $C = 1/2\pi\cdot\det(G) = 1/[2\pi\cdot\sigma_X\cdot\sigma_Y\cdot(1-\rho^2)^{1/2}]$
  and Q(u,v) is given by
  $[(u/\sigma_X)^2 - 2\rho\cdot(u/\sigma_X)\cdot(v/\sigma_Y) + (v/\sigma_Y)^2]/2(1-\rho^2)$
- The joint pdf of **X** and **Y** is given by
  $C\cdot\exp(-Q(u-\mu_X, v-\mu_Y))$
  where C and Q(u,v) are as defined above

Minor nitpick

- We derived the joint pdf of **X** and **Y** from the joint pdf of the independent unit Gaussian RVs **W** and **Z** assuming that the matrix G was invertible
- If G is not invertible, then
  a**W** + b**Z** = e•(c**W** + d**Z**)
  for some constant e
- In this case, the random variables **X** and **Y** are not jointly continuous

Formal definition — Preamble

- **X** and **Y** are said to be jointly Gaussian random variables with means μ_X and μ_Y respectively, variances $(\sigma_X)^2$ and $(\sigma_Y)^2$ respectively, and correlation coefficient ρ if

EITHER

(i) **X** is $\mathbf{N}(\mu_X, (\sigma_X)^2)$ and
Y = $\pm(\sigma_Y/\sigma_X)\mathbf{X} + (\mu_Y - \mu_X)$ is $\mathbf{N}(\mu_Y, (\sigma_Y)^2)$
(in this case, ρ = ±1 necessarily)

OR

Formal definition — the good stuff

(ii) the joint pdf of **X** and **Y** is

$C \cdot \exp(-Q(u-\mu_X, v-\mu_Y))$

where $C = 1/[2\pi \cdot \sigma_X \cdot \sigma_Y \cdot (1-\rho^2)^{1/2}]$

and Q(u,v) is given by

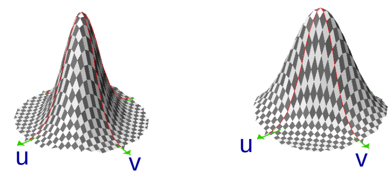
$[(u/\sigma_X)^2 - 2\rho \cdot (u/\sigma_X) \cdot (v/\sigma_Y) + (v/\sigma_Y)^2]/2(1-\rho^2)$

(in which case $|\rho| < 1$ necessarily)

- In either case, the marginal pdfs of **X** and **Y** are the Gaussian pdfs $\mathbf{N}(\mu_X, (\sigma_X)^2)$ and $\mathbf{N}(\mu_Y, (\sigma_Y)^2)$ respectively

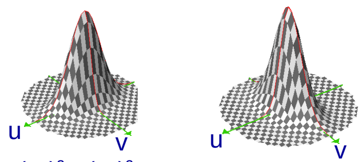
2 pictures = 2000 words: Part II

- **X**, **Y** are $\mathbf{N}(0, \sigma^2)$ with $0 < |\rho| < 1$



2 pictures = 2000 words: Part I redux

- **X**, **Y** independent zero-mean Gaussians with variances $(\sigma_X)^2$ and $(\sigma_Y)^2$



Jointly Gaussian jointly continuous

- If jointly Gaussian random variables **X** and **Y** are also jointly continuous, then their correlation coefficient satisfies $|\rho| < 1$, and their joint pdf is $C \cdot \exp(-Q(u-\mu_X, v-\mu_Y))$ where $C = 1/[2\pi \cdot \sigma_X \cdot \sigma_Y \cdot (1-\rho^2)^{1/2}]$ and Q(u,v) is given by
$[(u/\sigma_X)^2 - 2\rho \cdot (u/\sigma_X) \cdot (v/\sigma_Y) + (v/\sigma_Y)^2]/2(1-\rho^2)$
- Perfectly correlated (jointly Gaussian) RVs are not jointly continuous and cannot be described in terms of a joint pdf

Properties — I

- If **X** and **Y** are jointly Gaussian random variables with means μ_X and μ_Y respectively, variances $(\sigma_X)^2$ and $(\sigma_Y)^2$ respectively, and correlation coefficient ρ, then their marginal pdfs are also Gaussian
- **X** and **Y** are $\mathbf{N}(\mu_X, (\sigma_X)^2)$ and $\mathbf{N}(\mu_Y, (\sigma_Y)^2)$
- Note that the marginal pdfs do not depend on the value of the correlation coefficient
- All of the above holds even if $|\rho| = 1$

Properties — II

- If $\rho = 0$, then C and Q(u,v) simplify to $C = 1/[2\pi \cdot \sigma_X \cdot \sigma_Y]$ and $Q(u,v) = [(u/\sigma_X)^2 + (v/\sigma_Y)^2]/2$ and we get $f_{X,Y}(u,v) = f_X(u) \cdot f_Y(v)$
$= C \cdot \exp\{-([(u-\mu_X)/\sigma_X]^2 + [(v-\mu_Y)/\sigma_Y]^2)/2\}$
- Moral: Uncorrelated jointly Gaussian random variables are independent
- In general, uncorrelated random variables are not independent, but if they are jointly Gaussian, then they are also independent

Properties — III

- For jointly Gaussian random variables, "uncorrelatedness" is the same as independence
- The adjective jointly Gaussian is important here: the joint pdf must be jointly Gaussian
- If jointly continuous random variables have Gaussian marginal pdfs, then mustn't their joint pdf be the jointly Gaussian pdf?
- No

Marginally but not jointly Gaussian

- Value of the marginal pdf is the area of the cross-section of the joint pdf surface
- Let $f_{X,Y}(u,v) = (1/\pi)\bullet\exp[-(u^2 + v^2)/2]$ if $u \geq 0$ and $v \geq 0$, or if $u < 0$ and $v < 0$
- Comparison to independent **N**(0,1)s: mass in 2nd, 4th quadrants to 1st, 3rd

no mass in 2nd quadrant	double mass in 1st quadrant
double mass in 3rd quadrant	no mass in fourth quadrant

Double or nothing!

- Value of the marginal pdf is the area of the cross-section of the joint pdf surface
- In comparison to independent **N**(0,1)s, double cross-sectional area in one quadrant and none in other marginals are **N**(0,1) but joint pdf ≠ jointly Gaussian

no mass in 2nd quadrant	double mass in 1st quadrant
double mass in 3rd quadrant	no mass in fourth quadrant

Another look at the joint pdf

- We can obtain jointly Gaussian RVs **X** and **Y** starting from unit independent Gaussian RVs **W** and **Z** and setting

  $$(\mathbf{X}, \mathbf{Y}) = (\mathbf{W}, \mathbf{Z})G + (\mu_X, \mu_Y)$$

  where G is a 2×2 matrix
- But what should we choose G to be if we want **X** and **Y** to have specified variances $(\sigma_X)^2$ and $(\sigma_Y)^2$ respectively, and correlation coefficient ρ?

Non-unique answer

- $\mathbf{X} = a\mathbf{W} + b\mathbf{Z} + \mu_X$, $\mathbf{Y} = c\mathbf{W} + d\mathbf{Z} + \mu_Y$
- $(\sigma_X)^2 = \text{var}(\mathbf{X}) = a^2 + b^2$
- $(\sigma_Y)^2 = \text{var}(\mathbf{Y}) = c^2 + d^2$
- $\text{cov}(\mathbf{X}, \mathbf{Y}) = \rho\bullet\sigma_X\bullet\sigma_Y = ac + bd$
- Draw concentric circles of radiuses σ_X, σ_Y
- (a,b) and (c,d) are any pair of points on the circle that subtend an angle $\theta = \arccos(\rho)$ at the center. For example,
- $a = \sigma_X$, $b = 0$, $c = \sigma_Y\bullet\cos\theta$, $d = \sigma_Y\bullet\sin\theta$

What next?

- OK, so we have found a, b, c, and d, such that $\mathbf{X} = a\mathbf{W} + b\mathbf{Z} + \mu_X$, $\mathbf{Y} = c\mathbf{W} + d\mathbf{Z} + \mu_Y$ have the desired variances $(\sigma_X)^2$, $(\sigma_Y)^2$, and correlation coefficient ρ
- Next, to find the joint pdf, we set up G, take its inverse, …
- No, No, No, … a thousand times No!
- We know the form of the joint pdf: just plug in the values of μ_X, μ_Y, $(\sigma_X)^2$, $(\sigma_Y)^2$ and ρ

Sums of jointly Gaussian RVs — I

- If **X** and **Y** are jointly Gaussian random variables, then so are $\mathbf{A} = \alpha\mathbf{X} + \beta\mathbf{Y} + \gamma$ and $\mathbf{B} = \delta\mathbf{X} + \varepsilon\mathbf{Y} + \xi$ jointly Gaussian random variables
- We have seen that (**X**, **Y**) can be obtained from the independent pair (**W**,**Z**) of unit Gaussians by a linear transformation
- (**A**, **B**) is obtained from composite linear transformation (**W**,**Z**) → (**X**, **Y**) → (**A**, **B**)

Sums of jointly Gaussian RVs — II

- (**A**, **B**) is obtained from composite linear transformation (**W**,**Z**) → (**X**, **Y**) → (**A**, **B**)
- (**A**, **B**) is obtained from (**W**,**Z**) by linear transformation (**A**, **B**) jointly Gaussian
- To find the joint pdf of **A** and **B**, never, ever do it by matrices
- Figure out the means, variances, etc and plug into the standard form of the pdf!!

Sums of jointly Gaussian RVs — III

- $\mathbf{A} = \alpha\mathbf{X} + \beta\mathbf{Y} + \gamma$ and $\mathbf{B} = \delta\mathbf{X} + \varepsilon\mathbf{Y} + \xi$
- $E[\mathbf{A}] = E[\alpha\mathbf{X} + \beta\mathbf{Y} + \gamma] = \alpha\mu_X + \beta\mu_Y + \gamma$
- $E[\mathbf{B}] = E[\delta\mathbf{X} + \varepsilon\mathbf{Y} + \xi] = \delta\mu_X + \varepsilon\mu_Y + \xi$
- $\mathrm{var}(\mathbf{A}) = \alpha^2(\sigma_X)^2 + \beta^2(\sigma_Y)^2 + 2\alpha\beta\rho\sigma_X\sigma_Y$
- $\mathrm{var}(\mathbf{B}) = \delta^2(\sigma_X)^2 + \varepsilon^2(\sigma_Y)^2 + 2\delta\varepsilon\rho\sigma_X\sigma_Y$
- $\mathrm{cov}(\mathbf{A}, \mathbf{B})$
  $= \alpha\delta(\sigma_X)^2 + \beta\varepsilon(\sigma_Y)^2 + (\alpha\varepsilon + \beta\delta)\rho\sigma_X\sigma_Y$
  which gives the correlation coefficient, and we can thus <u>write down</u> the joint pdf

n jointly Gaussian RVs — I

- The joint pdf of two jointly Gaussian RVs is
  $C\bullet\exp(-Q(u-\mu_X, v-\mu_Y))$
  where $C = 1/[2\pi\bullet\sigma_X\bullet\sigma_Y\bullet(1-\rho^2)^{1/2}]$ and Q(u,v) is given by
  $[(u/\sigma_X)^2 - 2\rho\bullet(u/\sigma_X)\bullet(v/\sigma_Y) + (v/\sigma_Y)^2]/2(1-\rho^2)$
- Both quantities can be expressed in terms of the covariance matrix R of **X** and **Y**
- $R = \begin{pmatrix} (\sigma_X)^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & (\sigma_Y)^2 \end{pmatrix}$

n jointly Gaussian RVs — II

- $R = \begin{pmatrix} (\sigma_X)^2 & \rho\sigma_X\sigma_Y \\ \rho\sigma_X\sigma_Y & (\sigma_Y)^2 \end{pmatrix}$
- $C = 1/[2\pi\bullet\sigma_X\bullet\sigma_Y\bullet(1-\rho^2)^{1/2}]$
  $= 1/[((2\pi)^{1/2})^2\bullet\{\det(R)\}^{1/2}]$
- $Q(u,v) = (1/2)\bullet\{[u, v]\bullet R^{-1}\bullet[u, v]^T\}$
  $= (1/2)\bullet\underline{u}\bullet R^{-1}\bullet\underline{u}^T$ where $\underline{u} = [u,v]$
- $f_{X,Y}(u,v)$ can be expressed as
  $f_{\underline{X}}(\underline{u}) = C\bullet\exp\{-(1/2)\bullet(\underline{u}-\underline{\mu})\bullet R^{-1}\bullet(\underline{u}-\underline{\mu})^T\}$
  where $\underline{u} = [u,v]$ and $\underline{\mu} = [\mu_X, \mu_Y]$

n jointly Gaussian RVs — III

- n random variables $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n$ forming the vector $\underline{\mathbf{X}} = [\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n]$ are called jointly Gaussian (and $\underline{\mathbf{X}}$ is called a Gaussian vector) if their joint pdf is
  $f_{\underline{X}}(\underline{u}) = C\bullet\exp\{-Q(\underline{u}-\underline{\mu})\}$
  $= C\bullet\exp\{-(1/2)\bullet(\underline{u}-\underline{\mu})\bullet R^{-1}\bullet(\underline{u}-\underline{\mu})^T\}$
  where $C = 1/[(2\pi)^{n/2}\bullet\{\det(R)\}^{1/2}]$
  $Q(\underline{u}) = (1/2)\bullet\underline{u}\bullet R^{-1}\bullet\underline{u}^T$, $\underline{\mu} = E[\underline{\mathbf{X}}]$, and
  R is the covariance matrix $E[(\underline{\mathbf{X}}-\underline{\mu})^T\bullet(\underline{\mathbf{X}}-\underline{\mu})]$

n jointly Gaussian RVs — IV

- $\underline{\mathbf{X}}$ is called a Gaussian vector
- Nitpicking: $\underline{\mathbf{X}}$ is also called a Gaussian vector if it can be obtained by a singular linear transformation from m < n independent Gaussian random variables
- In this case, $\underline{\mathbf{X}}$ is not jointly continuous and does not possess a n-dimensional joint pdf
- We shall ignore this case which can be handled in terms of m RVs

Linear transformations again!

- If $\underline{\mathbf{X}}$ is a Gaussian vector, then $\underline{\mathbf{Y}} = \underline{\mathbf{X}}G$ is also a Gaussian vector with mean vector $\underline{\mu}G$ and covariance matrix $G^T \cdot R \cdot G$
- If R and G are nonsingular matrices, then the joint pdf of $\underline{\mathbf{Y}}$ can be written down
- Moral: linear transformations of jointly Gaussian RVs yield jointly Gaussian RVs
- You can do lots by working merely with covariance matrices

Summary — I

- The distribution of jointly Gaussian RVs (or a Gaussian vector) depends only on the mean vector $\underline{\mu}$ and the covariance matrix $R = E[(\underline{\mathbf{X}}-\underline{\mu})^T \cdot (\underline{\mathbf{X}}-\underline{\mu})]$
- If not jointly continuous, R is singular and $\underline{\mathbf{X}} = \underline{\mathbf{W}}G$ where $\underline{\mathbf{W}}$ is a vector of m < n independent Gaussian variables
- If jointly continuous, the joint pdf is $f_{\underline{X}}(\underline{u}) = C \cdot \exp\{-(1/2)\cdot(\underline{u}-\underline{\mu})\cdot R^{-1}\cdot(\underline{u}-\underline{\mu})^T\}$ where $C = 1/[(2\pi)^{n/2} \cdot \{\det(R)\}^{1/2}]$

Summary — II

- The marginal pdf of $\mathbf{X}_i$ is Gaussian with mean μ_i and variance $R_{i,i}$
- The marginal pdfs do not depend on the values of the off-diagonal entries in R
- Two jointly Gaussian jointly continuous RVs **X** and **Y** have joint pdf given by
  $$C \cdot \exp(-Q(u-\mu_X, v-\mu_Y))$$
  where $C = 1/[2\pi \cdot \sigma_X \cdot \sigma_Y \cdot (1-\rho^2)^{1/2}]$ and
  $Q(u,v) =$
  $[(u/\sigma_X)^2 - 2\rho \cdot (u/\sigma_X) \cdot (v/\sigma_Y) + (v/\sigma_Y)^2]/2(1-\rho^2)$

Summary — III

- For any random vector $\underline{\mathbf{X}}$, the random vector $\underline{\mathbf{Y}} = \underline{\mathbf{X}}G$ has mean vector $\underline{\mu}G$ and covariance matrix $G^T \cdot R \cdot G$
- If $\underline{\mathbf{X}}$ is a Gaussian vector, then $\underline{\mathbf{Y}} = \underline{\mathbf{X}}G$ is also a Gaussian vector (with mean vector $\underline{\mu}G$ and covariance matrix $G^T \cdot R \cdot G$)
- The joint pdf of $\underline{\mathbf{Y}}$ can be written down from this information since it depends only on $\underline{\mu}G$ and the covariance matrix $G^T \cdot R \cdot G$

