Two functions of two RVs

- **W** = g(**X**, **Y**) and **Z** = h(**X**, **Y**) are functions of random variables **X** and **Y**
- We have learned how to compute the pmf or pdf or CDF of **W** and **Z** individually
- What is the joint distribution of **W** and **Z**?
- The joint CDF $F_{W,Z}(\alpha,\beta)$ cannot be obtained from knowledge of the marginal CDFs $F_W(\alpha)$ and $F_Z(\beta)$
  … unless, of course, **W** and **Z** are known to be independent random variables

Two functions of two discrete RVs

- If **X** and **Y** are discrete random variables, then so are **W** = g(**X**, **Y**) and **Z** = h(**X**, **Y**)
- Determine the sets of values taken on by **W** and **Z**
- $\{\alpha_k\} = \{g(u_i, v_j) : 1 \le i \le m, 1 \le j \le n\}$
- $\{\beta_l\} = \{h(u_i, v_j) : 1 \le i \le m, 1 \le j \le n\}$
- $p_{W,Z}(\alpha_k, \beta_l) = P\{W = \alpha_k, Z = \beta_l\}$

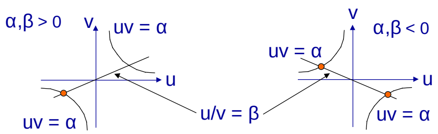
$$= \sum_i \sum_j p_{X,Y}(u_i,v_j)$$

such that $g(u_i, v_j) = \alpha_k$ and $h(u_i, v_j) = \beta_l$

Example M: W = XY, Z = X/Y

- If **W** = **XY** and **Z** = **X/Y**, then **W** and **Z** are either both positive or both negative
- Probability masses are in the 1st or 3rd quadrant of the plane with axes α and β
- For any α > 0, β > 0, **W** = α, **Z** = β
  if $X = +\sqrt{\alpha\beta}$, $Y = +\sqrt{\alpha/\beta}$
  or if $X = -\sqrt{\alpha\beta}$, $Y = -\sqrt{\alpha/\beta}$
- For any α < 0, β < 0, **W** = α, **Z** = β
  if $X = +\sqrt{\alpha\beta}$, $Y = -\sqrt{\alpha/\beta}$
  or if $X = -\sqrt{\alpha\beta}$, $Y = +\sqrt{\alpha/\beta}$

Example M: hyperbolic coordinates



- $p_{W,Z}(\alpha, \beta)$ = sum of the probability masses (if any) at the intersection of the hyperbola uv = α and the straight line u/v = β
- No intersection if sgn(α) ≠ sgn(β)

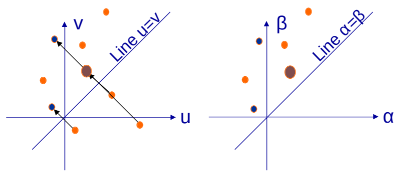
Example M: conclusion

- **W** = **XY** and **Z** = **X/Y** are both positive or both negative
- $p_{W,Z}(\alpha, \beta) = P\{X = +\sqrt{\alpha\beta}, Y = +\sqrt{\alpha/\beta}\}$
  $+ P\{X = -\sqrt{\alpha\beta}, Y = -\sqrt{\alpha/\beta}\}$
  $= p_{X,Y}(+\sqrt{\alpha\beta},+\sqrt{\alpha/\beta})$
  $+ p_{X,Y}(-\sqrt{\alpha\beta},-\sqrt{\alpha/\beta})$ if α > 0, β > 0
- Similarly, $p_{W,Z}(\alpha, \beta)$
  $= p_{X,Y}(+\sqrt{\alpha\beta},-\sqrt{\alpha/\beta})$
  $+ p_{X,Y}(-\sqrt{\alpha\beta},+\sqrt{\alpha/\beta})$ if α < 0, β < 0

Example N: min and max of two RV

- If **W** = min{**X**,**Y**} and **Z** = max{**X**,**Y**}, then **W** ≤ **Z** always
- Probability masses lie above the line α = β
- For any β ≥ α, **W** = α, **Z** = β
  if **X** = α, **Y** = β
  or if **X** = β, **Y** = α
- For any β < α, **W** = α, **Z** = β is impossible
- $p_{W,Z}(\alpha, \beta) = p_{X,Y}(\alpha, \beta) + p_{X,Y}(\beta, \alpha)$ if β ≥ α
- $p_{W,Z}(\alpha, \beta) = 0$ if β < α

The folding transformation



- Each mass below the line u = v folds over to its mirror image point (combining with any mass already present at the mirror image point)

Two functions of continuous RVs

- **X** and **Y** are jointly continuous random variables with joint pdf $f_{X,Y}(u,v)$
- g(u,v) and h(u,v) are continuous functions of u and v
- Usually, **W** = g(**X**, **Y**) and **Z** = h(**X**, **Y**) also are jointly continuous random variables
- However, this is not always true
- This (degenerate) case reduces to the problem of finding the pdf of a single function of two random variables

Example O: a degenerate case

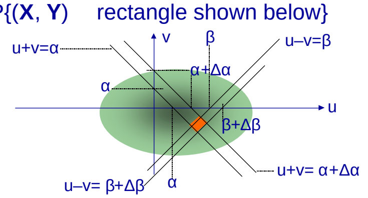
- **X** and **Y** are jointly continuous random variables with joint pdf $f_{X,Y}(u,v)$
- g(u,v) = cos(u+v); h(u,v) = sin(u+v)
- **W** = g(**X**, **Y**) and **Z** = h(**X**, **Y**)
- The random point (**W**, **Z**) is always on the circumference of the unit circle — which is a region of zero area
- Moral: **W** and **Z** are not jointly continuous random variables

Finding the joint pdf of W and Z

- For small values of Δα and Δβ,
  P{α ≤ **W** ≤ α+Δα, β ≤ **Z** ≤ β+Δβ}
  = P{(**W, Z**) small rectangular region}
  ≈ (joint pdf value)•(area of rectangle)
  = $f_{W,Z}(\alpha,\beta)$•Δα•Δβ
- If P{(**W, Z**) small rectangle}
  = P{(**X, Y**) some other small rectangle}
  then $f_{W,Z}(\alpha,\beta) = f_{X,Y}(u,v)$•(ratio of areas)
  where g(u,v) = α, h(u,v) = β

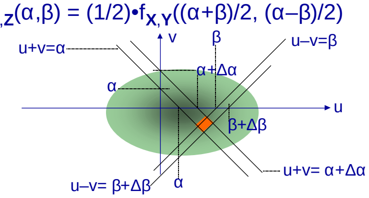
Example P: Sum and difference

- **W** = **X** + **Y** **Z** = **X** – **Y**
- P{α ≤ **W** ≤ α+Δα, β ≤ **Z** ≤ β+Δβ}
  = P{α ≤ **X** + **Y** ≤ α+Δα, β ≤ **X** – **Y** ≤ β+Δβ}
  = P{(**X**, **Y**) rectangle shown below}



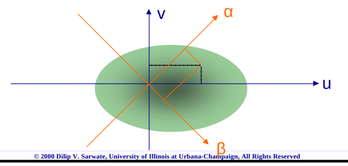
Example P: prob ≈ (joint pdf)•(area)

- P{α ≤ **X** + **Y** ≤ α+Δα, β ≤ **X** – **Y** ≤ β+Δβ}
  = P{(**X**, **Y**) rectangle shown below}
  ≈ $f_{X,Y}((\alpha+\beta)/2, (\alpha-\beta)/2)$•(Δα/√2)•(Δβ/√2)
- $f_{W,Z}(\alpha,\beta) = (1/2)$•$f_{X,Y}((\alpha+\beta)/2, (\alpha-\beta)/2)$



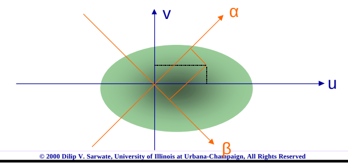
Example P: Rotation and scaling

- $f_{W,Z}(\alpha,\beta) = (1/2) \cdot f_{X,Y}((\alpha+\beta)/2, (\alpha-\beta)/2)$
- Surfaces represented by joint pdfs $f_{X,Y}$ and $f_{W,Z}$ are essentially similar: (note: factor of 2)
- The coordinate axes in the plane have rotated and scaled



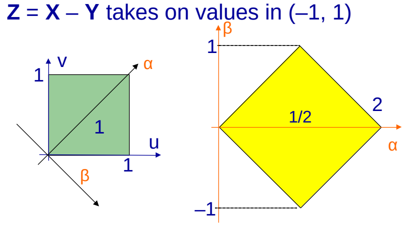
Example P: √2, √2, everywhere…

- Coordinates of the point shown are (u_0, v_0)
- Point has projections $(u_0+v_0)/\sqrt{2} = \alpha_0/\sqrt{2}$ and $(u_0-v_0)/\sqrt{2} = \beta_0/\sqrt{2}$ on the α-β axes
- Saying that point has coordinates (α_0, β_0) means α-β axes have been scaled by √2



Example P: support is different

- (**X**,**Y**) is uniformly distributed over square
- **W** = **X** + **Y** takes on values in (0 , 2)
- **Z** = **X** – **Y** takes on values in (–1, 1)



Example Q: min and max, again!

- **W** = min{**X**,**Y**}, **Z** = max{**X**,**Y**} **W** ≤ **Z**
- For β ≥ α and small values of Δα and Δβ,
  P{α ≤ **W** ≤ α+Δα, β ≤ **Z** ≤ β+Δβ}
  $= f_{W,Z}(\alpha,\beta) \cdot \Delta\alpha \cdot \Delta\beta$
  = P{α ≤ **X** ≤ α+Δα, β ≤ **Y** ≤ β+Δβ}
  + P{β ≤ **X** ≤ β+Δβ, α ≤ **Y** ≤ α+Δα}
  $= f_{X,Y}(\alpha,\beta) \cdot \Delta\alpha \cdot \Delta\beta + f_{X,Y}(\beta,\alpha) \cdot \Delta\beta \cdot \Delta\alpha$
- $f_{W,Z}(\alpha, \beta) = f_{X,Y}(\alpha, \beta) + f_{X,Y}(\beta, \alpha)$ if β ≥ α
- $f_{W,Z}(\alpha, \beta) = 0$ if β < α

Finding the joint pdf of W and Z

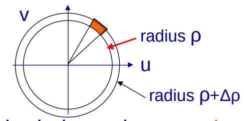
- For small values of Δα and Δβ,
  P{α ≤ **W** ≤ α+Δα, β ≤ **Z** ≤ β+Δβ}
  = P{(**W, Z**) small rectangular region}
  ≈ (joint pdf value)•(area of rectangle)
  $= f_{W,Z}(\alpha,\beta) \cdot \Delta\alpha \cdot \Delta\beta$
- If P{(**W, Z**) small rectangle}
  = P{(**X, Y**) some small area} then
  $f_{W,Z}(\alpha,\beta) = f_{X,Y}(u,v) \cdot$(ratio of areas)
  where g(u,v) = α, h(u,v) = β

Example R: Radius and angle

- **R** = $(\mathbf{X}^2 + \mathbf{Y}^2)^{1/2}$ is the distance of the random point (**X**, **Y**) from the origin
- Θ = arctan(**Y**/**X**) is the angle between the horizontal axis and the line through the origin and the random point (**X**, **Y**)
- P{$\rho \le$ **R** $\le \rho+\Delta\rho$, $\varphi \le \Theta \le \varphi+\Delta\varphi$}
  $\approx f_{\mathbf{R},\Theta}(\rho,\varphi)\bullet\Delta\rho\bullet\Delta\varphi$
  $\approx f_{\mathbf{X},\mathbf{Y}}(u,v)\bullet$(small area)
  where $u = \rho\bullet\cos(\varphi)$ and $v = \rho\bullet\sin(\varphi)$

Example R: What's the area?

- P{$\rho \le$ **R** $\le \rho+\Delta\rho$, $\varphi \le \Theta \le \varphi+\Delta\varphi$}
  $\approx f_{\mathbf{R},\Theta}(\rho,\varphi)\bullet\Delta\rho\bullet\Delta\varphi \approx f_{\mathbf{X},\mathbf{Y}}(u,v)\bullet$(small area)
  where $u = \rho\bullet\cos(\varphi)$ and $v = \rho\bullet\sin(\varphi)$



- The small shaded area is approximately a rectangle with sides $\rho\bullet\Delta\varphi$ (the arc-length) and $\Delta\rho$ area = $\rho\bullet\Delta\varphi\bullet\Delta\rho$

Example R: Joint pdf of R and Θ

- $f_{\mathbf{R},\Theta}(\rho,\varphi)\bullet\Delta\rho\bullet\Delta\varphi$
  $\approx f_{\mathbf{X},\mathbf{Y}}(u,v)\bullet$(small area)
  where $u = \rho\bullet\cos(\varphi)$ and $v = \rho\bullet\sin(\varphi)$
- But, small area = $\rho\bullet\Delta\varphi\bullet\Delta\rho$
- $f_{\mathbf{R},\Theta}(\rho,\varphi) = \rho\bullet f_{\mathbf{X},\mathbf{Y}}(\rho\bullet\cos(\varphi), \rho\bullet\sin(\varphi))$
  where $0 \le \rho < \infty$, $0 \le \varphi < 2\pi$
  … if **X** and **Y** take on values in the entire plane
- More complicated region otherwise

Circularly symmetric joint pdfs

- **X** and **Y** have joint pdf that is circularly symmetric about the origin if
  $f_{\mathbf{X},\mathbf{Y}}(u,v) = g(\rho)$ where $\rho = (u^2 + v^2)^{1/2}$
- $f_{\mathbf{R},\Theta}(\rho,\varphi) = \rho\bullet g(\rho)$, $0 \le \rho < \infty$, $0 \le \varphi < 2\pi$
- But, $f_{\mathbf{R}}(\rho) = 2\pi\bullet\rho\bullet g(\rho)$, for $\rho \ge 0$
- Look at the conditional pdf of Θ given that **R** = ρ
- $f_{\Theta|\mathbf{R}}(\varphi|\rho) = f_{\mathbf{R},\Theta}(\rho,\varphi)/f_{\mathbf{R}}(\rho) = 1/2\pi$, $0 \le \varphi < 2\pi$ is the same for all choices of ρ

Example R: R and Θ independent

- If **X** and **Y** have joint pdf that is circularly symmetric about the origin, then
- $f_{\mathbf{R},\Theta}(\rho,\varphi) = \rho\bullet g(\rho)$, $0 \le \rho < \infty$, $0 \le \varphi < 2\pi$
  $= [2\pi\bullet\rho\bullet g(\rho)]\bullet[1/2\pi]$
  $= f_{\mathbf{R}}(\rho)\bullet f_{\Theta}(\varphi)$
- **R** and Θ are independent RVs
- Θ is uniformly distributed on $[0, 2\pi)$
- **Y**/**X** = tan(Θ) has Cauchy pdf
- True for all circularly symmetric pdfs

Narrowband noise

- Output of narrowband filter whose input is thermal noise is modeled as
  $\mathbf{X}\bullet\cos(2\pi f_0 t) + \mathbf{Y}\bullet\sin(2\pi f_0 t)$
- **X** and **Y** are independent zero-mean Gaussian RVs with the same variance σ^2
- Amplitude of filter output is **R**, a Rayleigh random variable, and is independent of the phase Θ which is uniformly distributed on $[0, 2\pi)$. Noise power = $E[\mathbf{R}^2] = 2\sigma^2$

More general functions

- **W** = g(**X**, **Y**) and **Z** = h(**X**, **Y**)
- For small values of $\Delta\alpha$ and $\Delta\beta$, P{$\alpha \le$ **W** $\le \alpha+\Delta\alpha$, $\beta \le$ **Z** $\le \beta+\Delta\beta$}

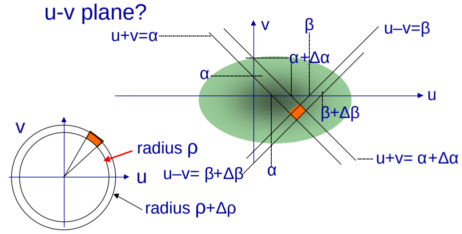
$\approx f_{W,Z}(\alpha,\beta)\bullet\Delta\alpha\bullet\Delta\beta$

$\approx f_{X,Y}(u,v)\bullet$(area of small region in u-v plane) where $g(u,v) = \alpha$, $h(u,v) = \beta$

- Small region is bounded by curves
  $g(u,v) = \alpha$, $h(u,v) = \beta$,
  $g(u,v) = \alpha+\Delta\alpha$, $h(u,v) = \beta+\Delta\beta$

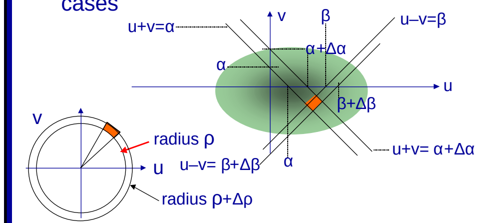
The $64,000 question

- What is the area of the small region in the u-v plane?



What's the difficulty?

- Curves do not intersect orthogonally in all cases



General answer to area question

- Area of the small region bounded by
  $g(u,v) = \alpha$, $h(u,v) = \beta$,
  $g(u,v) = \alpha+\Delta\alpha$, $h(u,v) = \beta+\Delta\beta$
  is $\Delta\alpha\bullet\Delta\beta$ divided by the absolute value of the Jacobian of the transformation
- $J(u, v) = \begin{vmatrix} \frac{\partial g(u,v)}{\partial u} & \frac{\partial g(u,v)}{\partial v} \\ \frac{\partial h(u,v)}{\partial u} & \frac{\partial h(u,v)}{\partial v} \end{vmatrix}$

$|J(u, v)| = |(\partial g/\partial u)\bullet(\partial h/\partial v) - (\partial g/\partial v)\bullet(\partial h/\partial u)|$

Is that your final answer?

- $f_{W,Z}(\alpha,\beta)\bullet\Delta\alpha\bullet\Delta\beta$
  $\approx f_{X,Y}(u,v)\bullet\Delta\alpha\bullet\Delta\beta/|J(u, v)|$
  where $g(u,v) = \alpha$, $h(u,v) = \beta$
- $f_{W,Z}(\alpha,\beta) = f_{X,Y}(u,v)/|J(u, v)|$
- How come the left side depends on α and β while the right side does not?
- For given α and β on the left side, u and v on the right are the numbers that satisfy
  $g(u,v) = \alpha$, $h(u,v) = \beta$

Tell me more, tell me more …

- $f_{W,Z}(\alpha,\beta) = f_{X,Y}(u,v)/|J(u, v)|$
- For given α and β on the left side, u and v on the right are the numbers that satisfy
  $g(u,v) = \alpha$, $h(u,v) = \beta$
- Take partial derivatives in the Jacobian
  $|J(u, v)| = |(\partial g/\partial u)\bullet(\partial h/\partial v) - (\partial g/\partial v)\bullet(\partial h/\partial u)|$
  and evaluate them at the numbers u and v satisfying $g(u,v) = \alpha$, $h(u,v) = \beta$
- RHS is a function of α and β too!

Example S: Doing it the simple way

- $\mathbf{W} = \mathbf{X} + \mathbf{Y}$ $\mathbf{Z} = \mathbf{X} - \mathbf{Y}$
- $g(u, v) = u+v$ $h(u, v) = u-v$
- $\partial g/\partial u = \partial g/\partial v = \partial h/\partial u = 1$; $\partial h/\partial v = -1$
- $J(u,v) = -2$
- If $g(u, v) = u+v = \alpha$ and $h(u, v) = u-v = \beta$, then $u = (\alpha+\beta)/2$ and $v = (\alpha-\beta)/2$
- $f_{\mathbf{W},\mathbf{Z}}(\alpha,\beta) = f_{\mathbf{X},\mathbf{Y}}(u,v)/|J(u,v)|$
- $f_{\mathbf{W},\mathbf{Z}}(\alpha,\beta) = f_{\mathbf{X},\mathbf{Y}}((\alpha+\beta)/2, (\alpha-\beta)/2)/2$ just as before

Example S: Radius and angle too!

- $\mathbf{R} = (\mathbf{X}^2 + \mathbf{Y}^2)^{1/2}$ $\Theta = \arctan(\mathbf{Y/X})$
- $g(u, v) = (u^2 + v^2)^{1/2}$; $h(u, v) = \arctan(v/u)$
- $\partial g/\partial u = u/(u^2 + v^2)^{1/2}$; $\partial g/\partial v = v/(u^2 + v^2)^{1/2}$
- $\partial h/\partial u = -v/(u^2 + v^2)$; $\partial h/\partial v = u/(u^2 + v^2)$
- $J(u,v) = 1/(u^2 + v^2)^{1/2} = 1/\rho$
- Given ρ and φ, $u = \rho\bullet\cos(\varphi)$; $v = \rho\bullet\sin(\varphi)$
- $f_{\mathbf{R},\Theta}(\rho,\varphi) = f_{\mathbf{X},\mathbf{Y}}(u,v)/|J(u, v)|$
- $f_{\mathbf{R},\Theta}(\rho,\varphi) = \rho\bullet f_{\mathbf{X},\mathbf{Y}}(\rho\bullet\cos(\varphi), \rho\bullet\sin(\varphi))$just as before

Transforming n random variables

- $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n$ are n random variables with joint (n-dimensional) pdf $f_{\underline{\mathbf{X}}}(\underline{u})$
- $\underline{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n)$
- $\underline{\mathbf{Y}} = (g_1(\underline{\mathbf{X}}), g_2(\underline{\mathbf{X}}), \ldots g_n(\underline{\mathbf{X}}))$ where the g_i are n functions of n real variables
- $f_{\underline{\mathbf{Y}}}(\underline{v}) = f_{\underline{\mathbf{X}}}(\underline{u})/|J(\underline{u})|$ where the i-jth entry in the Jacobian matrix is $\partial g_i/\partial u_j$
- On the right, $\underline{u}$ is the vector satisfying $\underline{v} = (g_1(\underline{u}), g_2(\underline{u}), \ldots g_n(\underline{u}))$ for the given $\underline{v}$

Example T: Linear transformations

- $\underline{\mathbf{Y}} = (g_1(\underline{\mathbf{X}}), g_2(\underline{\mathbf{X}}), \ldots g_n(\underline{\mathbf{X}}))$ where the g_i are linear functions of n real variables
- $\mathbf{Y}_k = g_{1,k}\mathbf{X}_1 + g_{2,k}\mathbf{X}_2 + \ldots g_{n,k}\mathbf{X}_n$, $1 \le k \le n$
- $\underline{\mathbf{Y}} = \underline{\mathbf{X}}G$ where G is an $n\times n$ matrix and we assume that G is nonsingular
- Jacobian of transformation is $\det(G) \neq 0$
- $f_{\underline{\mathbf{Y}}}(\underline{v}) = f_{\underline{\mathbf{X}}}(\underline{u})/|J(\underline{u})| = f_{\underline{\mathbf{X}}}(\underline{v}G^{-1})/|\det(G)|$
- This result will be used later

Sighs of relief …

- General transformations for multiple random variables not tested on exams etc. except for the following special cases dealt with in the examples noted:
  - sum and difference of two variables (P)
  - minima and maxima for n variables (Q)
  - radius and angle for two variables (R)
  - linear transformations of n variables (T)

Summary

- Methods for computing the joint pmf or pdf of two functions of two RVs were studied
- Problems studied include sum and difference, minimum and maximum, radius and angle
- For circularly symmetric pdfs, the radius and the angle are independent random variables; and angle is uniform on $[0, 2\pi)$