## ECE 313

### Probability with Engineering Applications

### Functions of Many Random Variables II

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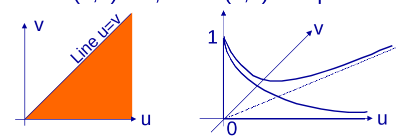
### Function of jointly continuous RVs

- **X** and **Y** are jointly continuous, and **Z** is continuous? Find the pdf of **Z** = g(**X**, **Y**) as follows:
  - Figure out where joint pdf is nonzero
  - Sketch the curve g(u, v) = α in u-v plane
  - Find $F_Z(\alpha)$ = P{**Z** ≤ α} = P{g(**X**, **Y**) ≤ α} = P{(**X**, **Y**)   region g(u, v) ≤ α in plane}
  - Repeat for other values of α
  - Differentiate $F_Z(\alpha)$ = P{**Z** ≤ α} with respect to α to find $f_Z(\alpha)$



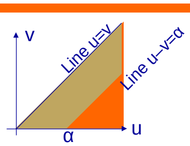
### Example C: Z = X – Y

- The joint pdf of **X** and **Y** is given by $f_{X,Y}(u,v) = \exp(-u)$ for $0 \le v \le u < \infty$ and $f_{X,Y}(u,v) = 0$ otherwise
- Find the pdf of **Z = X – Y**
- **X** is Γ(2,1) RV; **Y** is Γ(1,1) = exponential





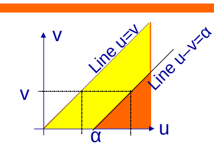

### Example C: P{Z ≤ α} = P{X – Y ≤ α}




- P{**X** – **Y** ≤ α} = volume under pdf surface in green shaded region
  = integral of $f_{X,Y}(u,v)$ over green shaded region
  = 0 if α < 0



### Example C: Finding P{X – Y ≤ α}



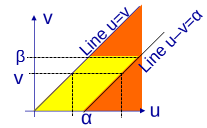

- For any fixed value of v, $0 \le v < \infty$, u varies from v to α+v
- $P\{X - Y \le \alpha\} = \int_{v=0}^{\infty} \int_{u=v}^{\alpha+v} \exp(-u)\, du\, dv$
  $= 1 - \exp(-\alpha)$



### Example C: pdf of Z = X – Y

- CDF of **Z** = **X** – **Y** is 1 – exp(–α), α > 0
- $f_Z(\alpha) = \exp(-\alpha)$, α > 0
- **Z** is an exponential RV with parameter 1
- **Z** is a gamma random variable with parameters (1,1), that is, **Z** is a Γ(1,1) RV
- Remember that **Y** is a Γ(1,1) RV
- If **Y** and **Z** were independent Γ(1,1) RVs, then their sum **Y** + **Z** would be a Γ(2,1) RV
- But, **Y** + **Z** = **X** is a Γ(2,1) RV
- So, are **Y** and **Z** independent Γ(1,1) RVs?

## Example C: Y & Z are independent




$P\{\mathbf{Y} \le \beta, \mathbf{Z} \le \alpha\} = \int_{v=0}^{\beta} \int_{u=v}^{\alpha+v} \exp(-u)\, du\, dv$

$= [1 - \exp(-\beta)] \cdot [1 - \exp(-\alpha)]$

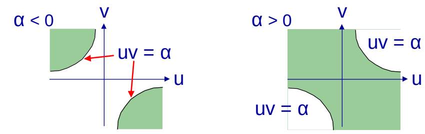
$= P\{\mathbf{Y} \le \beta\} \cdot P\{\mathbf{Z} \le \alpha\}$ since **Y**, **Z** are $\Gamma(1,1)$ RV

Hence, **Y** and **Z** are independent $\Gamma(1,1)$ RVs



## The function Z = XY

- Let **Z** = **XY**
- $P\{\mathbf{Z} \le \alpha\} = P\{\mathbf{XY} \le \alpha\}$ is the volume under the joint pdf in the shaded region below

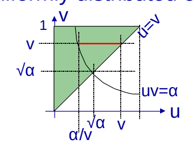



- If $\alpha > 0$, usually easier to find $P\{\mathbf{Z} > \alpha\}$



## Example D: P{Z > α} = P{XY > α}

- (**X**, **Y**) uniformly distributed on triangle



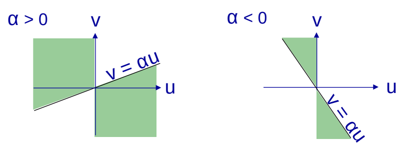

- $P\{\mathbf{XY} > \alpha\} = \int_{v=\sqrt{\alpha}}^{1} \int_{u=\alpha/v}^{v} 2\, du\, dv = \int_{v=\sqrt{\alpha}}^{1} 2v - \alpha/v\, dv$

  $= 1 - \alpha + \alpha \ln \alpha,\ 0 < \alpha < 1$
- $f_{\mathbf{Z}}(\alpha) = -\ln \alpha,\ 0 < \alpha < 1$



## The function Z = Y/X

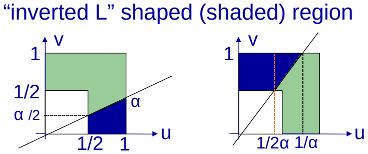
- Let **Z** = **Y/X**
- $P\{\mathbf{Z} \le \alpha\} = P\{\mathbf{Y/X} \le \alpha\} = P\{\mathbf{Y} \le \alpha\mathbf{X}\}$ is the volume under the joint pdf in the shaded region below






## Example E: P{Z ≤ α} = P{Y/X ≤ α}

- (**X**,**Y**) is uniformly distributed over the "inverted L" shaped (shaded) region




- If $0 \le \alpha \le 1$, $P\{\mathbf{Y/X} \le \alpha\} = \alpha/2$
- $f_{\mathbf{Z}}(\alpha) = 1/2$
- If $1 \le \alpha < \infty$, $P\{\mathbf{Y/X} \le \alpha\} = 1 - 1/2\alpha$
- $f_{\mathbf{Z}}(\alpha) = 1/2\alpha^2$



## Circularly symmetric joint pdfs

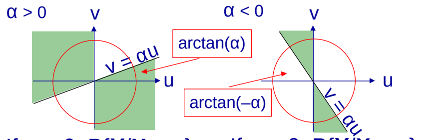
- **X** and **Y** are said to have a joint pdf that is circularly symmetric about the origin if the value of $f_{\mathbf{X,Y}}(u,v)$ depends only on

  $$\rho = (u^2 + v^2)^{1/2}$$

  and not on the individual values of u and v
- We write $f_{\mathbf{X,Y}}(u,v) = g((u^2 + v^2)^{1/2}) = g(\rho)$
- **Z** = **Y/X** has Cauchy pdf for all joint pdfs that are circularly symmetric about the origin

## Example F: Why a Cauchy pdf?

- A pizza slice subtending angle $\theta$ radians at the center will give you $\theta/2\pi$ of the pizza




- If $\alpha > 0$, $P\{\mathbf{Y}/\mathbf{X} \le \alpha\}$ $= [\pi + 2\bullet\arctan(\alpha)]/2\pi$
- If $\alpha < 0$, $P\{\mathbf{Y}/\mathbf{X} \le \alpha\}$ $= [\pi - 2\bullet\arctan(-\alpha)]/2\pi$
- $f_Z(\alpha) = 1/[\pi(1+\alpha^2)]$
- $f_Z(\alpha) = 1/[\pi(1+\alpha^2)]$



## Example G: What's your angle?

- **X** and **Y** have a circularly symmetric pdf, and hence, **Z** = **Y**/**X** has a Cauchy pdf
- $\theta = \arctan(\mathbf{Z})$ = angle that the line through (**X**, **Y**) and (0, 0) makes with the u axis
- For all **W**, $F_W(\mathbf{W})$ is a uniform RV on (0, 1)
- Since $F_Z(\alpha) = (1/2) + (1/\pi)\bullet\arctan(\alpha)$, $(1/2)+(1/\pi)\bullet\arctan(\mathbf{Z})$ is uniform on (0, 1)
- $\theta = \arctan(\mathbf{Z})$ is uniform RV on $(-\pi/2, \pi/2)$
- Intuitively obvious from symmetry



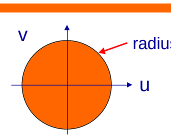
## The function $R = (X^2 + Y^2)^{1/2}$

- Let $\mathbf{R} = (\mathbf{X}^2 + \mathbf{Y}^2)^{1/2}$ be the distance of the random point (**X**, **Y**)from the origin
- $P\{\mathbf{R} \le \alpha\} = P\{(\mathbf{X}^2 + \mathbf{Y}^2)^{1/2} \le \alpha\}$ $= P\{(\mathbf{X}^2 + \mathbf{Y}^2) \le \alpha^2\}$

  is the volume under the joint pdf in a disk of radius $\alpha$ centered at the origin
- disk = interior of circle
- A switch from rectangular coordinates (u,v) to polar coordinates $(\rho, \theta)$ usually helps…



## Finding $P\{(X^2 + Y^2)^{1/2} \le \alpha\}$



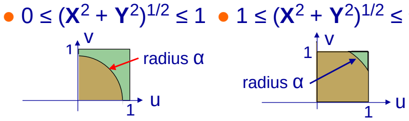

- $P\{(\mathbf{X}^2 + \mathbf{Y}^2)^{1/2} \le \alpha\} = \int\int_{\text{circle of radius } \alpha} f_{X,Y}(u,v)\, du\, dv$

$$= \int_{\rho=0}^{\alpha}\int_{\theta=0}^{2\pi} f_{X,Y}(\rho\cos\theta, \rho\sin\theta)\, \rho\bullet d\theta\, d\rho$$



## Example H: Minding the details

- $f_{X,Y}(u,v) = 2u$ for $0 \le u \le 1$, $0 \le v \le 1$
- Two cases need to be considered
- $0 \le (\mathbf{X}^2 + \mathbf{Y}^2)^{1/2} \le 1$
- $1 \le (\mathbf{X}^2 + \mathbf{Y}^2)^{1/2} \le \sqrt{2}$

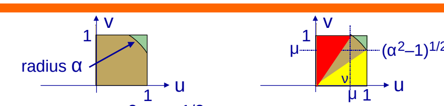



For $0 \le \alpha \le 1$,

$$P\{\mathbf{R} \le \alpha\} = \int_{\rho=0}^{\alpha}\int_{\theta=0}^{\pi/2} 2\bullet\rho\cos\theta\, \rho\bullet d\theta\, d\rho = 2\alpha^3/3$$



## Example H: Integration by parts?




- Let $\mu = (\alpha^2 - 1)^{1/2}$
- Let $\nu = \arctan(\mu)$

$$P\{\mathbf{R} \le \alpha\} = \int_{u=0}^{1}\int_{v=0}^{\mu u} 2u\bullet dv\bullet du + \int_{u=0}^{\mu}\int_{v=u/\mu}^{1} 2u\bullet dv\bullet du$$

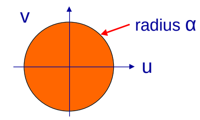
$$+ \int_{\rho=0}^{\alpha}\int_{\theta=\nu}^{\pi/2-\nu} 2\bullet\rho\cos\theta\, \rho\bullet d\theta\, d\rho = \alpha^2 - \frac{1}{3} - \frac{2}{3}\mu^3$$

## Example H: Putting it all together

- $P\{\mathbf{R} \le \alpha\} = P\{(\mathbf{X}^2 + \mathbf{Y}^2)^{1/2} \le \alpha\}$
  $= 2\alpha^3/3$ for $0 \le \alpha \le 1$
- $P\{\mathbf{R} \le \alpha\} = P\{(\mathbf{X}^2 + \mathbf{Y}^2)^{1/2} \le \alpha\}$
  $= \alpha^2 - 1/3 - (2/3)(\alpha^2 - 1)^{3/2}$ for $1 \le \alpha \le \sqrt{2}$
- $f_{\mathbf{R}}(\alpha) = 2\alpha^2$ for $0 \le \alpha \le 1$
- $f_{\mathbf{R}}(\alpha) = 2\alpha \bullet (1 - (\alpha^2 - 1)^{1/2})$ for $1 \le \alpha \le \sqrt{2}$
- $f_{\mathbf{R}}(\alpha) = 0$ otherwise



## Finding $P\{(X^2 + Y^2)^{1/2} \le \alpha\}$



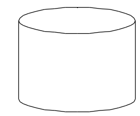

- $P\{(\mathbf{X}^2 + \mathbf{Y}^2)^{1/2} \le \alpha\} = \int\int_{\text{circle of radius } \alpha} f_{\mathbf{X},\mathbf{Y}}(u,v)\, du\, dv$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} f_{\mathbf{X},\mathbf{Y}}(\rho \cos\theta, \rho \sin\theta)\, \rho \bullet d\theta\, d\rho$$



## Example I: pdf = an inverted cone

- $f_{\mathbf{X},\mathbf{Y}}(u,v) = (3/2\pi) \bullet (u^2+v^2)^{1/2}$, $(u^2+v^2)^{1/2} < 1$





## General result for pdf of $(X^2 + Y^2)^{1/2}$

- When the joint pdf of ($\mathbf{X}$, $\mathbf{Y}$) has circular symmetry about the origin,
  $F_{\mathbf{R}}(\alpha) = P\{\mathbf{R} \le \alpha\} = P\{(\mathbf{X}^2 + \mathbf{Y}^2)^{1/2} \le \alpha\}$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} f_{\mathbf{X},\mathbf{Y}}(\rho \cos\theta, \rho \sin\theta)\, \rho \bullet d\theta\, d\rho$$

$$= \int_{\rho=0}^{\alpha} \int_{\theta=0}^{2\pi} g(\rho)\, \rho \bullet d\theta\, d\rho = 2\pi \int_{\rho=0}^{\alpha} \rho \bullet g(\rho)\, d\rho$$

- $f_{\mathbf{R}}(\alpha) = 2\pi \bullet \alpha \bullet g(\alpha)$, for $\alpha \ge 0$

## Example J: Gauss and Rayleigh

- $\mathbf{X}$ and $\mathbf{Y}$ are independent zero-mean Gaussian RVs with the same variance $\sigma^2$
- $f_{\mathbf{X},\mathbf{Y}}(u,v) = (1/2\pi\sigma^2) \bullet \exp[-(u^2 + v^2)/2\sigma^2]$
- $g(\rho) = (1/2\pi\sigma^2) \bullet \exp[-\rho^2/2\sigma^2]$
- $f_{\mathbf{R}}(\alpha) = (\alpha/\sigma^2) \bullet \exp[-\alpha^2/2\sigma^2]$, for $\alpha \ge 0$
- $\mathbf{R}$ is a Rayleigh random variable
- Reminder: Hazard rate of Rayleigh RV increases linearly with age



## Maxima and minima of RVs

- Let $\mathbf{Z} = \max\{\mathbf{X}, \mathbf{Y}\}$ be the larger of the values taken on by $\mathbf{X}$ and $\mathbf{Y}$ on the trial of the experiment
- Let $\mathbf{W} = \min\{\mathbf{X}, \mathbf{Y}\}$ be the smaller of the values taken on by $\mathbf{X}$ and $\mathbf{Y}$ on the trial of the experiment
- Given the joint CDF of $\mathbf{X}$ and $\mathbf{Y}$, what are the CDFs of $\mathbf{Z}$ and $\mathbf{W}$?
- pdfs or pmfs?

### The function Z = max{X, Y}

- **Z** = max{**X**, **Y**} $\le \beta$ if and only if both **X** and **Y** are smaller than $\beta$
- P{**Z** $\le \beta$} = P{**X** $\le \beta$, **Y** $\le \beta$}
- Remember: commas mean intersections
- $F_Z(\beta) = F_{X,Y}(\beta, \beta)$
- **X**, **Y** independent $\quad F_Z(\beta) = F_X(\beta)F_Y(\beta)$
- If **X**, **Y** are also jointly continuous, then $f_Z(\beta) = f_X(\beta)F_Y(\beta) + F_X(\beta)f_Y(\beta)$



### Example K: max of exponential RVs

- **X** and **Y** are independent exponential random variables with parameter $\lambda$
- $f_{\max\{X,Y\}}(\beta) = f_X(\beta)F_Y(\beta) + F_X(\beta)f_Y(\beta)$
  $= 2\cdot\lambda\cdot\exp(-\lambda\beta)\cdot[1 - \exp(-\lambda\beta)]$
  $= 2\cdot\lambda\cdot\exp(-\lambda\beta) - (2\lambda)\cdot\exp(-(2\lambda)\beta),\ \beta > 0$
- max{**X**, **Y**} is not a gamma RV
- $\lambda\cdot\exp(-\lambda\beta)$ is an exponential pdf
- $(2\lambda)\cdot\exp(-(2\lambda)\beta)$ is also exponential pdf



### Maximum of n random variables

- **Z** = max{$X_1, X_2, \ldots X_n$} $\le \beta \quad$ all $X_i \le \beta$
- P{**Z** $\le \beta$} = P{$X_1 \le \beta, X_2 \le \beta, \ldots, X_n \le \beta$}
- $F_Z(\beta) = F_{X_1, X_2, \ldots, X_n}(\beta, \beta, \ldots, \beta)$
- $X_i$'s independent $\quad F_Z(\beta) = \prod_{i=1}^{n} F_{X_i}(\beta)$
- If the $X_i$'s are also jointly continuous, then
  $$f_Z(\beta) = \prod_{i=1}^{n} F_{X_i}(\beta) \sum_{j=1}^{n} f_{X_j}(\beta)/F_{X_j}(\beta)$$



### More on the maximum of n RVs

- Special case: the $X_i$'s are independent jointly continuous random variables all with the same marginal CDF F(u) and pdf f(u)
- $F_{\max\{X_1, X_2, \ldots, X_n\}}(\beta) = [F(\beta)]^n$
- $f_{\max\{X_1, X_2, \ldots, X_n\}}(\beta) = n\cdot[F(\beta)]^{n-1}\cdot f(\beta)$
- Example: If the $X_i$'s are uniformly distributed on [0, 1], F(u) = u for $0 \le u \le 1$
- $f_{\max\{X_1, X_2, \ldots, X_n\}}(\beta) = n\cdot\beta^{n-1}$ for $0 \le \beta \le 1$



### The function W = min{X, Y}

- **W** = min{**X**, **Y**} > $\alpha \quad$ **X** > $\alpha$ and also **Y** > $\alpha$
- P{**W** > $\alpha$} = P{**X** > $\alpha$, **Y** > $\alpha$}
- But, $P(A^c \cap B^c) = P((A\ \ B)^c) = 1 - P(A\ \ B)$
- $1 - F_W(\alpha) = 1 - [F_X(\alpha) + F_Y(\alpha) - F_{X,Y}(\alpha,\alpha)]$
- If **X**, **Y** are independent random variables, then $[1 - F_W(\alpha)] = [1 - F_X(\alpha)]\cdot[1 - F_Y(\alpha)]$
- If **X**, **Y** are also jointly continuous, then $f_W(\alpha) = f_X(\alpha)\cdot[1 - F_Y(\alpha)] + f_Y(\alpha)\cdot[1 - F_X(\alpha)]$



### Example L: min of exponential RVs

- **X** and **Y** are independent exponential random variables with parameters $\lambda_X, \lambda_Y$
- $f_{\min\{X,Y\}}(\alpha) = f_X(\alpha)[1-F_Y(\alpha)] + [1-F_X(\alpha)]f_Y(\alpha)$
  $= \lambda_X\cdot\exp(-\lambda_X\alpha)\cdot\exp(-\lambda_Y\alpha)$
  $+ \lambda_Y\cdot\exp(-\lambda_Y\alpha)\cdot\exp(-\lambda_X\alpha)$
  $= (\lambda_X + \lambda_Y)\cdot\exp(-(\lambda_X + \lambda_Y)\cdot\alpha),\ \alpha > 0$
- In contrast to max{**X**, **Y**}, min{**X**, **Y**} is an exponential RV with parameter $\lambda_X + \lambda_Y$

## Minimum of n random variables

- $\mathbf{W} = \min\{\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n\} > \alpha$ all $\mathbf{X}_i > \alpha$
- $P\{\mathbf{W} > \alpha\} = P\{\mathbf{X}_1 > \alpha, \mathbf{X}_2 > \alpha, \ldots, \mathbf{X}_n > \alpha\}$
- If $\mathbf{X}_i$'s are independent random variables then $[1 - F_{\mathbf{W}}(\alpha)] = \prod_{i=1}^{n} [1 - F_{\mathbf{X}_i}(\alpha)]$
- If the $\mathbf{X}_i$'s are also jointly continuous, then $f_{\mathbf{W}}(\alpha) = \prod_{i=1}^{n} [1 - F_{\mathbf{X}_i}(\alpha)] \sum_{j=1}^{n} f_{\mathbf{X}_j}(\alpha)/[1 - F_{\mathbf{X}_j}(\alpha)]$



## min of many exponential RVs

- $\mathbf{X}_i$'s are independent exponential random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$
  $1 - F_{\mathbf{X}_i}(\alpha) = \exp(-\lambda_i \alpha)$
  $1 - F_{\min\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\}}(\alpha) = \prod [1 - F_{\mathbf{X}_i}(\alpha)]$
  $= \exp\left[-\left(\sum \lambda_i\right)\alpha\right]$
- Hence, $\min\{\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n\}$ is an exponential RV with parameter $\sum \lambda_i$



## More on the minimum of n RVs

- Special case: the $\mathbf{X}_i$'s are independent jointly continuous random variables all with the same marginal CDF F(u) and pdf f(u)
- $1 - F_{\min\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\}}(\alpha) = [1 - F(\alpha)]^n$
- $f_{\min\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\}}(\alpha) = n \cdot [1 - F(\alpha)]^{n-1} \cdot f(\alpha)$
- Example: If the $\mathbf{X}_i$'s are uniformly distributed on [0, 1], F(u) = u for u   [0, 1]
- $f_{\min\{\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n\}}(\alpha) = n \cdot (1-\alpha)^{n-1}$, $\alpha$   [0, 1]



## System lifetimes and hazard rates

- The positive RVs $\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n$ represent the lifetimes of n components of a system
- Suppose that the system fails as soon as one component fails
- Lifetime of system: $\mathbf{W} = \min\{\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n\}$
- Suppose that failures are independent
- Then, $[1 - F_{\mathbf{W}}(\alpha)] = \prod_{i=1}^{n} [1 - F_{\mathbf{X}_i}(\alpha)]$
- What is the hazard rate of $\mathbf{W}$?



## CDF in terms of hazard rate

- $1 - F_{\mathbf{W}}(\alpha) = \exp\left(-\int_{t=0}^{\alpha} h_{\mathbf{W}}(t)\,dt\right)$
- $1 - F_{\mathbf{X}_i}(\alpha) = \exp\left(-\int_{t=0}^{\alpha} h_{\mathbf{X}_i}(t)\,dt\right)$
- But, $[1 - F_{\mathbf{W}}(\alpha)] = \prod_{i=1}^{n} [1 - F_{\mathbf{X}_i}(\alpha)]$
- $1 - F_{\mathbf{W}}(\alpha) = \exp\left(-\int_{t=0}^{\alpha} \sum_{i=0}^{n} h_{\mathbf{X}_i}(t)\,dt\right)$



## Hazard rate of the minimum

- $\mathbf{W} = \min\{\mathbf{X}_1, \mathbf{X}_2, \ldots \mathbf{X}_n\}$ where the $\mathbf{X}_i$'s are independent positive random variables
- $h_{\mathbf{W}}(t) = \sum_{i=0}^{n} h_{\mathbf{X}_i}(t)$
- Hazard rate of the system is the sum of the hazard rates of the components
- Hazard rate of the system exceeds the largest of hazard rates of the components

## Hazard rate, schmazard rate…

- Hazard rate large average lifetime small
- Expected lifetime of system is smaller than expected lifetime of the worst component
- Special case: $\mathbf{X}_1$, $\mathbf{X}_2$, … $\mathbf{X}_n$ ~ exponential RVs with parameters $\lambda_i$ and $E[\mathbf{X}_i] = (\lambda_i)^{-1}$
- $\mathbf{W}$ ~ exponential RV with parameter $\sum \lambda_i$
- $E[\mathbf{W}] = (\sum \lambda_i)^{-1} << (\max \lambda_i)^{-1} = \min E[\mathbf{X}_i]$
- $\lambda_i = \lambda$ for all i $E[\mathbf{W}] = 1/n\lambda = E[\mathbf{X}]/n$



## Summary — I

- We found the pdf of functions of RVs
  - Difference $\mathbf{X} - \mathbf{Y}$
  - Product $\mathbf{XY}$
  - Ratio $\mathbf{Y/X}$
  - Angle arctan($\mathbf{Y/X}$)
  - Radius vector $\mathbf{R} = (\mathbf{X}^2 + \mathbf{Y}^2)^{1/2}$
  - Max{$\mathbf{X}$, $\mathbf{Y}$} and Min{$\mathbf{X}$,$\mathbf{Y}$}
  - Max{$\mathbf{X}_1$, $\mathbf{X}_2$,…,$\mathbf{X}_n$} and Min{$\mathbf{X}_1$, $\mathbf{X}_2$,…$\mathbf{X}_n$}



## Summary — II

- In all cases, we found the CDF of the function of RVs and deduced the pdf
- For all joint pdfs with circularly symmetry about the origin, $\mathbf{Y/X}$ has Cauchy pdf and arctan($\mathbf{Y/X}$) is uniformly distributed on $(-\pi/2, \pi/2)$
- If the joint pdf is circularly symmetric about the origin, then $(\mathbf{X}^2 + \mathbf{Y}^2)^{1/2}$ has pdf $2\pi\cdot\alpha\cdot f_{X,Y}(\alpha \cos\theta, \alpha \sin\theta) = 2\pi\cdot\alpha\cdot g(\alpha)$



## Summary — III

- The complementary CDF $1 - F_{\mathbf{W}}(\alpha)$ of the minimum of independent random variables is the product of the complementary CDFs of the random variables
- The minimum of independent exponential RVs with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ is an exponential RV with parameter $\sum \lambda_i$
- Hazard rate of minimum of independent RVs is the sum of the hazard rates



## ECE 313

## Probability with Engineering Applications

### Functions of Many Random Variables II

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