Random point in the plane

- (**X**,**Y**) denotes a random point in the plane
- The random variable **Z** is a function of **X** and **Y**; say **Z** = g(**X**,**Y**)
- Example: **Z** = **X** + **Y** meaning that outcome ω Ω is mapped onto the number **Z**(ω) = **X**(ω) + **Y**(ω) by **Z**
- Questions:
  - Discrete RV **Z**: What is the pmf of **Z**?
  - Continuous RV **Z**: What is the pdf of **Z**?

Discrete random variable Z = g(X,Y)

- **X** and **Y** are discrete RVs taking on values $u_1, u_2, \ldots, u_n, \ldots$ and $v_1, v_2, \ldots, v_m, \ldots$ respectively
- **Z** is a discrete RV taking on values in set

  $\{g(u_i, v_j), 1 \le i \le n, 1 \le j \le m\}$
- The m×n values need not all be distinct
- $p_Z(\alpha) = P\{\mathbf{Z} = \alpha\} = \sum_i \sum_j p_{X,Y}(u_i,v_j)$

  such that $g(u_i, v_j) = \alpha$

Discrete random variable Z = g(X,Y)

- **Z** has value α at all points on the "curve" g(u,v) = α in the u-v plane
- Simply add up all the probability masses that happen to lie on the "curve" to get

  $p_Z(\alpha) = P\{\mathbf{Z} = \alpha\} = \sum_i \sum_j p_{X,Y}(u_i,v_j)$

  such that $g(u_i, v_j) = \alpha$
- Repeat for all choices of values of **Z**
- Check that you have found a valid pmf

Example: Sum: Z = X + Y

- $p_Z(\alpha) = P\{\mathbf{Z} = \alpha\} = \sum_i p_{X,Y}(u_i, \alpha - u_i)$
- An important special case is when **X** and **Y** take on integer values only
- $p_Z(n) = P\{\mathbf{Z} = n\} = \sum_i p_{X,Y}(i, n - i)$
- A similar result applies whenever the values of **X** and **Y** are equally spaced along the axes

Sum of integer-valued discrete RVs

- $p_Z(n) = P\{\mathbf{Z} = n\} = \sum p_{X,Y}(i, n - i)$
- $p_{X,Y}(i, n - i) = p_{Y|X}(n-i|i) \bullet p_X(i) = q(n-i) \bullet p_X(i)$
- $p_Z(n) = P\{\mathbf{Z} = n\} = p_{X+Y}(n) = \sum q(n-i) \bullet p_X(i)$

  = discrete convolution of q(•) and $p_X(\bullet)$, the unconditional pmf of **X**
- Conditioned on **X** = i, P{**X** + **Y** = n|**X** = i} = P{**Y** = n–i|**X** = i}
- The convolution shown obtains P{**Z** = n} via the theorem of total probability

Sum of independent discrete RVs

- $p_Z(n) = P\{\mathbf{Z} = n\} = \sum p_{X,Y}(i, n - i)$
- If **X** and **Y** are independent discrete RVs, then $p_{Y|X}(n-i|i) = p_Y(n-i)$ for all choices of i
- $p_{X,Y}(u,v) = p_X(u) \bullet p_Y(v)$ for all u and v
- $p_Z(n) = P\{\mathbf{Z} = n\} = \sum p_X(i) \bullet p_Y(n - i)$

  = discrete convolution of pmfs of **X** and **Y**
- Moral: If **X** and **Y** are independent discrete RVs taking on integer values, then $p_Z = p_{X+Y} = p_X \ast p_Y$

Sum of independent binomial RVs

- If **X** and **Y** are independent discrete RVs taking on integer values, then $p_Z = p_X \ast p_Y$
- If **X** and **Y** are independent binomial RVs with parameters (n, p) and (m, p), then **X** + **Y** is a binomial RV with parameters (n+m, p): Note that p is the same for both
- **X** is # of successes in n independent trials **Y** is # of successes in next m trials
- Independent trials independent RVs
- **X** + **Y** is # of successes in n+m trials

Independent negative binomial RVs

- If **X** and **Y** are independent negative binomial RVs with parameters (n, p) and (m, p), then **X** + **Y** is a negative binomial RV with parameters (n+m, p) Note that p is the same for both
- **X** is waiting time for n successes **Y** is waiting time for next m successes
- Independent trials independent RVs
- **X** + **Y** is waiting time for n+m successes
- First wait for n; then wait for additional m

Sum of independent Poisson RVs

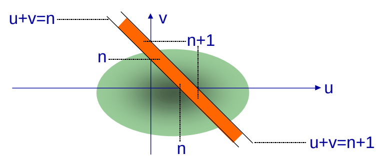
- If **X** and **Y** are independent discrete RVs taking on integer values, then $p_Z = p_X \ast p_Y$
- If **X** and **Y** are independent Poisson RVs with parameters λ and μ, then **X** + **Y** is a Poisson RV with parameter λ + μ
- Poisson process with arrival rate 1
- **X** is # of arrivals in time interval (0,λ] **Y** is # of arrivals in time interval (λ, λ + μ]
- Disjoint intervals **X** and **Y** independent
- **X** + **Y** is # of arrivals in (0, λ + μ]

Sums of independent discrete RVs

- The mathematical details of the proofs of these assertions are in the text
- More important that you understand why the results hold from first principles
- More important that you remember the results and apply them where necessary
- Results generalize to sums of n binomial or Poisson or negative binomial RVs
- Parameter of sum is sum of parameters
- Same p for binomial and negative binomial

X, Y continuous but Z is discrete

- When **X** and **Y** are jointly continuous RVs, it is possible that **Z** is a discrete RV
- Example: **Z** = ë**X** + **Y**û has integer values



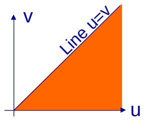
- P{**Z** = n} = probability mass/volume in shaded orange region

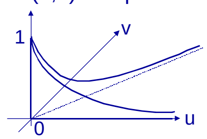
Function of jointly continuous RVs

- **X** and **Y** are jointly continuous, and **Z** is continuous? Find the pdf of **Z** = g(**X**, **Y**) as follows:
  - Figure out where joint pdf is nonzero
  - Sketch the curve g(u, v) = α in u-v plane
  - Find $F_Z(\alpha)$ = P{**Z** ≤ α} = P{g(**X**, **Y**) ≤ α} = P{(**X**, **Y**) region g(u, v) ≤ α in plane}
  - Repeat for other values of α
  - Differentiate $F_Z(\alpha)$ = P{**Z** ≤ α} with respect to α to find $f_Z(\alpha)$

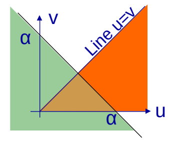
Example A: Z = X + Y

- The joint pdf of **X** and **Y** is given by $f_{X,Y}(u,v) = \exp(-u)$ for $0 \le v \le u < \infty$ and $f_{X,Y}(u,v) = 0$ otherwise
- Find the pdf of **Z** = **X** + **Y**
- **X** is Γ(2,1) RV; **Y** is Γ(1,1) = exponential



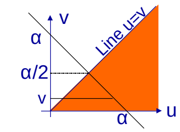


Example A: P{Z ≤ α} = P{X + Y ≤ α}



- P{**X** + **Y** ≤ α} = volume under pdf surface in green shaded region
  = integral of $f_{X,Y}(u,v)$ over green shaded region
  = 0 if α < 0

Example A: Finding P{X + Y ≤ α}

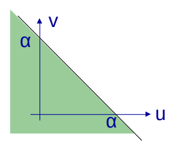


- For any fixed value of v, 0 ≤ v ≤ α/2, u varies from v to α–v
- $P\{X + Y \le \alpha\} = \int_{v=0}^{\alpha/2} \int_{u=v}^{\alpha-v} \exp(-u)\, du\, dv$
  $= 1-2\bullet\exp(-\alpha/2)+\exp(-\alpha) = (1 - \exp(-\alpha))^2$

CDF and pdf of Z

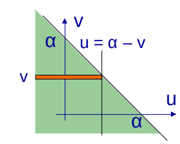
- CDF of **Z** = **X** + **Y** is $(1 - \exp(-\alpha))^2$, α > 0
- Is this valid?
- Yes; continuous monotone increasing function that is 0 at α = 0 and that approaches 1 as α → ∞
- $f_Z(\alpha) = 2\bullet(1 - \exp(-\alpha))\bullet\exp(-\alpha)$
  $= 2\bullet\exp(-\alpha) - 2\bullet\exp(-2\alpha)$, α > 0
- Note that this is not a gamma pdf
- 2•exp(–2α) is Γ(2,1) pdf
- exp(–α) is Γ(1,1) pdf

Sum of jointly continuous RVs — I



- P{**X** + **Y** ≤ α} = volume under pdf surface in green shaded region
  = integral of $f_{X,Y}(u,v)$ over green shaded region

Sum of jointly continuous RVs — II



fixed value of v; u varies from –∞ to α – v

Then let v vary from –∞ to ∞

- $P\{X + Y \le \alpha\} = \int_{v=-\infty}^{\infty} \int_{u=-\infty}^{\alpha-v} f_{X,Y}(u,v)\, du\, dv$
- $P\{X + Y \le \alpha\} = \int_{u=-\infty}^{\infty} \int_{v=-\infty}^{\alpha-u} f_{X,Y}(u,v)\, dv\, du$

Sum of jointly continuous RVs — III

- $P\{\mathbf{X}+\mathbf{Y}\le\alpha\} = \int_{v=-\infty}^{\infty}\int_{u=-\infty}^{\alpha-v} f_{\mathbf{X},\mathbf{Y}}(u,v)\,du\,dv$
- $f_{\mathbf{Z}}(\alpha)$ = derivative of $F_{\mathbf{Z}}(\alpha)$ with respect to α
  = derivative of $P\{\mathbf{X}+\mathbf{Y}\le\alpha\}$ w.r.t. α
  = derivative of double integral above with respect to α
- We will need to use the magic formula for differentiating an integral (from Lecture 32) two times to compute this derivative

Derivative of an integral

- $\int_{a(\alpha)}^{b(\alpha)} g(x;\alpha)\,dx = G(\alpha)$
- What is the derivative of the integral, that is, the function $G(\alpha)$, with respect to α?
- $\frac{\partial}{\partial\alpha}G(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}g(x;\alpha)\,dx$
  $+ g(b(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}b(\alpha)$
  $- g(a(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}a(\alpha)$

Derivative of integral: special cases

- Special case: The limits do not depend on α; only the integrand depends on α
- $\frac{\partial}{\partial\alpha}G(\alpha) = \int_{a}^{b} \frac{\partial}{\partial\alpha}g(x;\alpha)\,dx$
- Special case: Only the limits depend on α; the integrand does not depend on α
- $\frac{\partial}{\partial\alpha}G(\alpha)$
  $= g(b(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}b(\alpha) - g(a(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}a(\alpha)$

Sum of jointly continuous RVs — IV

- $P\{\mathbf{X}+\mathbf{Y}\le\alpha\} = \int_{v=-\infty}^{\infty}\int_{u=-\infty}^{\alpha-v} f_{\mathbf{X},\mathbf{Y}}(u,v)\,du\,dv$
- $P\{\mathbf{X}+\mathbf{Y}\le\alpha\} = \int_{v=-\infty}^{\infty} h(v;\alpha)\,dv$ where $h(v;\alpha)$ is the inner integral with respect to u
- $\frac{\partial}{\partial\alpha}G(\alpha) = \int_{a}^{b} \frac{\partial}{\partial\alpha}g(x;\alpha)\,dx$
- $\frac{\partial}{\partial\alpha}P\{\mathbf{X}+\mathbf{Y}\le\alpha\} = \int_{v=-\infty}^{\infty} \frac{\partial}{\partial\alpha}h(v;\alpha)\,dv$

Sum of jointly continuous RVs — V

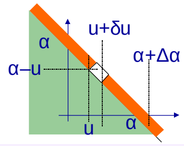
- $\frac{\partial}{\partial\alpha}P\{\mathbf{X}+\mathbf{Y}\le\alpha\} = \int_{v=-\infty}^{\infty} \frac{\partial}{\partial\alpha}h(v;\alpha)\,dv$
- $\frac{\partial}{\partial\alpha}h(v;\alpha) = \frac{\partial}{\partial\alpha}\int_{u=-\infty}^{\alpha-v} f_{\mathbf{X},\mathbf{Y}}(u,v)\,du$
- $\frac{\partial}{\partial\alpha}G(\alpha)$
  $= g(b(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}b(\alpha) - g(a(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}a(\alpha)$
- $\frac{\partial}{\partial\alpha}h(v;\alpha) = f_{\mathbf{X},\mathbf{Y}}(\alpha-v,v)\bullet\frac{\partial}{\partial\alpha}(\alpha-v)$
- $\frac{\partial}{\partial\alpha}P\{\mathbf{X}+\mathbf{Y}\le\alpha\} = \int_{v=-\infty}^{\infty} f_{\mathbf{X},\mathbf{Y}}(\alpha-v,v)\,dv$

Sum of jointly continuous RVs — VI

- $f_{\mathbf{Z}}(\alpha)$ = derivative of $F_{\mathbf{Z}}(\alpha)$ with respect to α
  = derivative of $P\{\mathbf{X}+\mathbf{Y}\le\alpha\}$ w.r.t. α
- $f_{\mathbf{Z}}(\alpha) = \frac{\partial}{\partial\alpha}P\{\mathbf{X}+\mathbf{Y}\le\alpha\} = \int_{v=-\infty}^{\infty} f_{\mathbf{X},\mathbf{Y}}(\alpha-v,v)\,dv$
- Alternative double integral for $P\{\mathbf{X}+\mathbf{Y}\le\alpha\}$ can also be differentiated with respect to α
- $f_{\mathbf{Z}}(\alpha) = \frac{\partial}{\partial\alpha}P\{\mathbf{X}+\mathbf{Y}\le\alpha\} = \int_{u=-\infty}^{\infty} f_{\mathbf{X},\mathbf{Y}}(u,\alpha-u)\,du$

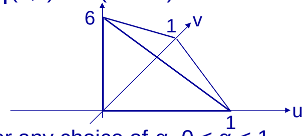
Sum of jointly continuous RVs – VII

- $f_{X+Y}(\alpha) = \int_{u=-\infty}^{\infty} f_{X,Y}(u,\alpha-u)\,du$
- $P\{\alpha \le \mathbf{X} + \mathbf{Y} \le \alpha+\Delta\alpha\}$ = volume in strip
  = sum of volumes in small rectangles
  = sum of $f_{X,Y}(u,\alpha-u)\bullet(\delta u\bullet\sqrt{2})\bullet(\Delta\alpha/\sqrt{2})$



Example B

- $f_{X,Y}(u,v) = 6\bullet(1-u-v)$ for $0 \le u \le 1-v \le 1$



- For any choice of α, $0 \le \alpha \le 1$,
  $f_{X,Y}(u, \alpha-u) = 6\bullet(1-u-(\alpha-u)) = 6\bullet(1-\alpha)$
  if $0 \le u \le 1-(\alpha-u) \le 1$, i.e., $0 \le u \le \alpha$
- $f_{X+Y}(\alpha) = \int_{u=0}^{\alpha} f_{X,Y}(u,\alpha-u)\,du = 6\bullet(1-\alpha)\bullet\alpha$

Sum of independent RVs — I

- $f_{X+Y}(\alpha) = \int_{u=-\infty}^{\infty} f_{X,Y}(u,\alpha-u)\,du$
- If **X** and **Y** are independent random variables, then $f_{X,Y}(u,v) = f_X(u)\bullet f_Y(v)$ for all u and v
- $f_{X+Y}(\alpha) = \int_{u=-\infty}^{\infty} f_X(u)\bullet f_Y(\alpha-u)\,du$
- Moral: the pdf of the sum of independent jointly continuous random variables is the convolution of their marginal pdfs

Sum of independent RVs — II

- Ross obtains the following result for independent jointly continuous RVs
- $F_{X+Y}(\alpha) = \int_{u=-\infty}^{\infty} f_X(u)\bullet F_Y(\alpha-u)\,du$
- He then states that F_{X+Y} is the convolution of the marginal CDFs F_X and F_Y
- This is incorrect
- In fact, F_{X+Y} is the convolution of the marginal pdf f_X and the marginal CDF F_Y

Sum of independent gamma RVs: I

- If **X** and **Y** are independent RVs, then $f_{X+Y} = f_X \circledast f_Y$
- If **X** and **Y** are independent gamma RVs with parameters (s, λ) and (t, λ), then **X** + **Y** is a gamma RV with parameters $(s+t, \lambda)$
- Note that the scale parameter λ must be the same, but the order parameters can be different
- The order parameters add; scale is same

Sum of independent gamma RVs: II

- If **X** and **Y** are independent $\Gamma(s, \lambda)$ and $\Gamma(t, \lambda)$ RVs, then **X** + **Y** is a $\Gamma(s+t, \lambda)$ RV
- Assume s and t are integers
- **X** is the waiting time for s arrivals in a Poisson process of intensity λ
- **Y** is the waiting time for the next t arrivals
- Disjoint intervals **X** and **Y** independent
- **X** + **Y** is the waiting time for s+t arrivals
- Result applies even when s and t are not integers (but no nice interpretation…)

Sum of independent gamma RVs: III

- If **X** and **Y** are independent $\Gamma(s, \lambda)$ and $\Gamma(t, \lambda)$ RVs, then **X** + **Y** is a $\Gamma(s+t, \lambda)$ RV
- More generally, if $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are independent $\Gamma(t_i, \lambda)$ RVs, then

  $\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n$ is a $\Gamma\left(\sum t_i, \lambda\right)$ RV
- Independence is important
- In Example A, **X** is a $\Gamma(2,1)$ RV and **Y** is a $\Gamma(1,1)$, but they are not independent; and their sum is not a gamma random variable

Sum of independent gamma RVs: IV

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are independent $\Gamma(t_i, \lambda)$, then $\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n$ is a $\Gamma\left(\sum t_i, \lambda\right)$ RV
- Let $\mathbf{Y}_1, \mathbf{Y}_2, \ldots, \mathbf{Y}_n$ denote independent unit Gaussian RVs, and let $\mathbf{X}_i = (\mathbf{Y}_i)^2$, $1 \le i \le n$
- $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are independent $\Gamma(1/2, 1/2)$ RVs, also known as chi-squared (χ^2) RVs with one degree of freedom
- $\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n$ is a $\Gamma(n/2, 1/2)$ RV, also called a χ^2 RV with n degrees of freedom

Sum of independent Gaussian RVs

- If $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are independent $\mathbf{N}(\mu_i, (\sigma_i)^2)$ RVs, then $\mathbf{X}_1 + \mathbf{X}_2 + \ldots + \mathbf{X}_n$ is a $\mathbf{N}\left(\sum \mu_i, \sum (\sigma_i)^2\right)$ RV
- The mean and variance of a sum of independent Gaussian RVs are the sums of the means and variances respectively
- Actually, the mean of a sum of arbitrary RVs is the sums of the means — neither independence nor Gaussianity is required

Mean and variance

- The mean of a sum of arbitrary RVs is the sums of the means
- The variance of a sum of independent RVs is the sum of the variances — Gaussianity is not required for this result to hold
- Example: **X** and **Y** are independent $\Gamma(s, \lambda)$ and $\Gamma(t, \lambda)$ RVs with means s/λ, t/λ and variances s/λ^2, t/λ^2 respectively
- **X** + **Y** is $\Gamma(s+t, \lambda)$ with mean $(s+t)/\lambda$; variance $(s+t)/\lambda^2$

Characteristic functions — I

- pdf/pmf of the sum of independent RVs is the convolution of their marginal pdfs/pmfs
- Fourier transform methods can be used to compute convolutions of pdfs/pmfs in the “frequency domain”
- The characteristic function of the RV **X** is

$$\Phi_{\mathbf{X}}(\omega) = E[\exp(j\omega\mathbf{X})] = \sum \exp(j\omega u_i) \cdot p_{\mathbf{X}}(u_i)$$

$$\text{or} \int_{u=-\infty}^{\infty} \exp(j\omega u) \cdot f_{\mathbf{X}}(u)\, du$$

Characteristic functions — II

- The characteristic function of the RV **X** is

$$\Phi_{\mathbf{X}}(\omega) = E[\exp(j\omega\mathbf{X})] = \sum \exp(j\omega u_i) \cdot p_{\mathbf{X}}(u_i)$$

$$\text{or} \int_{u=-\infty}^{\infty} \exp(j\omega u) \cdot f_{\mathbf{X}}(u)\, du$$

- $f_{\mathbf{X}}(u) = (1/2\pi) \int_{\omega=-\infty}^{\infty} \exp(-j\omega u) \cdot \Phi_{\mathbf{X}}(\omega)\, d\omega$
- For independent RVs **X** and **Y**, $\Phi_{\mathbf{X}+\mathbf{Y}}(\omega) = \Phi_{\mathbf{X}}(\omega) \cdot \Phi_{\mathbf{Y}}(\omega)$

Characteristic functions — III

- Characteristic functions are treated in Chapter 7 of the text
- See also the related notion of the moment-generating function
- Neither of these topics will be discussed further in this course
- Nor are they required material
- But, you may use these concepts freely in solving problems on homework and exams

Summary — I

- The notion of a function of multiple random variables was introduced
- Problems involving functions of discrete random variables are easy to solve
- pmf of a sum of independent discrete RVs is the convolution of the marginal pmfs
- Special cases: binomial RVs, negative binomial RVs, and Poisson RVs

Summary — II

- The general method of computing the pdf of a function of jointly continuous RVs was introduced
- The pdf of a sum of two RVs can be expressed as an integral of the joint pdf
- For independent RVs, this integral is a convolution integral
- Special cases: gamma and Gaussian RVs

Summary — III

- The mean of a sum of RVs is the sum of the means of the RVs
- The variance of a sum of independent RVs is the sum of the variances of the RVs
- For the special case of independent Gaussian RVs, the sum is also a Gaussian RV, and the above results allow us to write down the pdf of the sum