Conditional pdf of X — Introduction

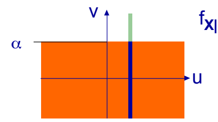
- For a continuous random variable **X**, the conditional pdf $f_{X|A}(u|A)$ describes the probabilistic behavior of **X** given that the event A has occurred
- $P\{u \le X \le u+\Delta u \mid A\}$
  $= P[\{u \le X \le u+\Delta u\} \cap A]/P(A) \approx f_{X|A}(u|A) \cdot \Delta u$
- In Lecture 29, we considered the case when A is specified in terms of **X** itself

Conditional pdf of X — given {Y ≤ α}

- **X** and **Y** are jointly continuous random variables, and suppose $A = \{Y \le \alpha\}$
- Given that A occurred, (**X**,**Y**) must be in the orange shaded region shown

$P\{u \le X \le u+\Delta u\}$ = volume in green strip

$P[\{u \le X \le u+\Delta u\} \cap A]$ = volume in blue strip

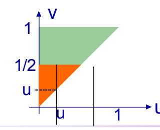


$$f_{X|A}(u|A) \cdot \Delta u \approx \frac{\text{vol. blue strip}}{P(A)}$$

$$f_{X|A}(u|A) = \frac{\int_{-\infty}^{\alpha} f_{X,Y}(u,v)dv}{F_Y(\alpha)}$$

Example D: conditional pdf given A

- The random point (**X**,**Y**) is uniformly distributed on region $\{(u,v): 0 < u < v < 1\}$
- $f_{X,Y}(u,v) = 2$ for $0 < u < v < 1$; green region
- $A = \{Y \le 1/2\}$. $P(A) = 1/4$; orange region

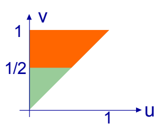


$$f_{X|A}(u|A) = \frac{\int_{-\infty}^{1/2} f_{X,Y}(u,v)dv}{P(A)}$$

$f_{X|A}(u|A) = 2 \cdot (1/2 - u)/(1/4)$
$= 4(1-2u)$ for $0 < u < 1/2$
and 0 otherwise

Example D: conditional pdf given A^c

- The random point (**X**,**Y**) is uniformly distributed on region $\{(u,v): 0 < u < v < 1\}$
- $f_{X,Y}(u,v) = 2$ for $0 < u < v < 1$; green region
- $A^c = \{Y > 1/2\}$. $P(A^c) = 3/4$; orange region

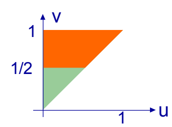


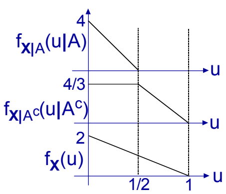
$$f_{X|A^c}(u|A^c) = \frac{\int_{1/2}^{\infty} f_{X,Y}(u,v)dv}{P(A^c)}$$

$= 4/3$ for $0 < u < 1/2$
$= 8/3(1-u)$ for $1/2 \le u < 1$
$= 0$ otherwise

Example D: unconditional pdf

- $f_{X,Y}(u,v) = 2$ for $0 < u < v < 1$; green region
- $f_X(u) = f_{X|A}(u|A)P(A) + f_{X|A^c}(u|A^c)P(A^c)$
- $A^c = \{Y > 1/2\}$. $P(A^c) = 3/4$; orange region





Conditional pdf of X: given {Y = α}

- Suppose $A = \{Y = \alpha\}$ instead of $\{Y \le \alpha\}$
- Now what is $f_{X|A}(u|A)$?
- The conditioning event A has probability zero, so the usual definition will not work!
- Nonetheless, we have observed event A
- We wish to understand the probabilistic behavior of **X** under these circumstances
- We use a limiting argument to obtain the conditional pdf

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Conditional pdf of X: derivation

- Assume first that $A = \{\alpha \le \mathbf{Y} \le \alpha+\Delta\alpha\}$
- $f_{X|A}(u|A) \cdot \Delta u \approx \dfrac{P[\{u \le \mathbf{X} \le u+\Delta u\} \cap A]}{P(A)}$
- $f_{X|A}(u|A) \cdot \Delta u \approx \dfrac{f_{X,Y}(u,\alpha) \cdot \Delta u \cdot \Delta\alpha}{f_Y(\alpha) \cdot \Delta\alpha}$
- In the limit as $\Delta\alpha \to 0$, the event $A = \{\alpha \le \mathbf{Y} \le \alpha+\Delta\alpha\}$ approaches $\{\mathbf{Y} = \alpha\}$
- $f_{X|Y}(u|\alpha) = \dfrac{f_{X,Y}(u,\alpha)}{f_Y(\alpha)}$ provided $f_Y(\alpha) > 0$

Conditional pdf of X: definition

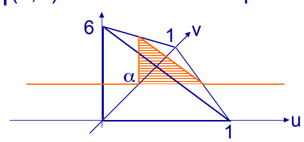
- The conditional pdf of **X** given that **Y** = α is
  $f_{X|Y}(u|\alpha) = \dfrac{f_{X,Y}(u,\alpha)}{f_Y(\alpha)}$ provided $f_Y(\alpha) > 0$
- $f_{X|Y}(u|\alpha)$ is a function of u
- α is a parameter; it is the numerical value of **Y** on this trial; $f_Y(\alpha)$ is just a number
- If $f_Y(\alpha) = 0$, $f_{X|Y}(u|\alpha)$ is defined to be zero
- What does the conditional pdf look like?

Conditional pdf of X: more notions

- The conditional pdf of **X** given that **Y** = α is
  $f_{X|Y}(u|\alpha) = \dfrac{f_{X,Y}(u,\alpha)}{f_Y(\alpha)}$ provided $f_Y(\alpha) > 0$
- $f_{X|Y}(u|\alpha)$ is a function of u; α is a parameter
- $f_Y(\alpha)$ is a constant
- The conditional pdf of **X** given that **Y** = α is just the function $f_{X,Y}(u,\alpha)$ multiplied by a scaling factor $1/f_Y(\alpha)$
- $f_{X,Y}(u,\alpha)$ = cross-section of pdf surface!

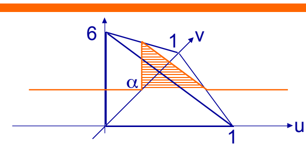
Conditional pdf of X — geometry

- $f_{X,Y}(u,\alpha)$ = cross-section of pdf surface!



- Why do we need a scaling factor of $1/f_Y(\alpha)$ to get the conditional pdf?
- Area under $f_{X,Y}(u,\alpha) \ne 1$ in general
- Area = $f_Y(\alpha) \Rightarrow f_{X,Y}(u,\alpha)/f_Y(\alpha)$ is a valid pdf

Example E: Conditional pdf of X



- $f_{X,Y}(u,v) = 6 \cdot (1-u-v)$ for $0 \le u \le 1-v \le 1$
- $f_{X,Y}(u,\alpha) = 6 \cdot (1-\alpha-u)$ for $0 \le u \le 1-\alpha$
- Area under curve $= (1/2) \cdot [6 \cdot (1-\alpha)] \cdot (1-\alpha) = 3 \cdot (1-\alpha)^2 = f_Y(\alpha)$, as obtained previously
- $f_{X|Y}(u|\alpha) = 2 \cdot (1-\alpha-u)/(1-\alpha)^2$ for $0 \le u \le 1-\alpha$

The theorem of total probability…

- The conditional pdf of **X** given that **Y** = α is
  $f_{X|Y}(u|\alpha) = \dfrac{f_{X,Y}(u,\alpha)}{f_Y(\alpha)}$ provided $f_Y(\alpha) > 0$
- $f_X(u)$, the unconditional (i.e. marginal) pdf of **X**, is found by "integrating out the unwanted variable" from the joint pdf
- $f_X(u) = \displaystyle\int_{v=-\infty}^{\infty} f_{X,Y}(u,v)\, dv$
- $f_X(u) = \displaystyle\int_{\alpha=-\infty}^{\infty} f_{X,Y}(u,\alpha)\, d\alpha = \int_{\alpha=-\infty}^{\infty} f_{X|Y}(u|\alpha) f_Y(\alpha)\, d\alpha$

… in integral form

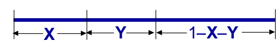
- The conditional pdf of **X** given that **Y** = α is $f_{X|Y}(u|\alpha) = \frac{f_{X,Y}(u,\alpha)}{f_Y(\alpha)}$ provided $f_Y(\alpha) > 0$
- $f_X(u) = \int_{\alpha=-\infty}^{\infty} f_{X,Y}(u,\alpha)\,d\alpha = \int_{\alpha=-\infty}^{\infty} f_{X|Y}(u|\alpha) f_Y(\alpha)\,d\alpha$
- Compare $p_X(u_i) = \sum p_{X|Y}(u_i|v_j) \cdot p_Y(v_j)$
- This is just the theorem of total probability

$$P(B) = \sum P(B|A_j)P(A_j)$$

expressed in terms of integrals and pdfs

Example F: Breaking a stick

- A stick of unit length is broken at a random point **X**
- The right-hand piece (which is of length 1–**X**) is broken at **Y**
- **Y** is the distance of the breakpoint from the left end of the right-hand piece
- The stick is broken into three pieces of lengths **X**, **Y**, and 1–**X**–**Y**

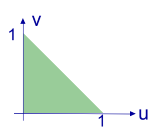


Example F: choosing breakpoints

- The first breakpoint **X** is chosen at random
- **X** is uniformly distributed on (0,1)
- The second breakpoint **Y** (on the right-hand piece, which is of length 1–**X**) is chosen at random on the piece
- **Y** is the distance of the breakpoint from the left end of the right-hand piece
- Conditioned on **X** = α, **Y** is uniformly distributed on the interval $(0, 1-\alpha)$

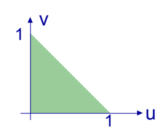
Example F: joint pdf of X and Y

- **X** and **Y** take on values in the interval (0,1)
- It is always true that **Y** < 1–**X**
- $f_{X,Y}(u,v)$, the joint pdf of **X** and **Y**, is nonzero only in the region $0 < v < 1-u < 1$



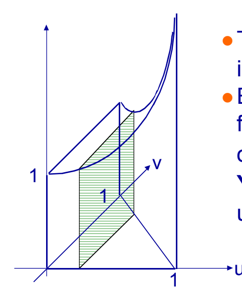
What's the joint pdf of X and Y?

- **X** is uniformly distributed on (0,1)
- $f_X(u) = 1$ for $0 < u < 1$, and 0 elsewhere
- Given that **X** = α, the conditional pdf of **Y** is uniform on $(0, 1-\alpha)$
- $f_{Y|X}(v|\alpha) = 1/(1-\alpha)$ for $0 < v < 1-\alpha$



- $f_{X,Y}(u,v) = 1/(1-u)$ for $0 < v < 1-u < 1$
- As $u \to 1$, $f_{X,Y}(u,v) \to \infty$

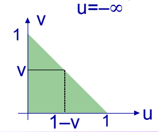
Example F: the joint pdf surface



- The joint pdf surface is as shown
- Each cross-section is $f_{Y|X}(v|\alpha)$ = rectangle of area 1; **Y** is conditionally uniform given **X**

Example F: marginal pdf of Y

- Conditioned on **X** = α, **Y** is uniformly distributed on the interval $(0,1-\alpha)$
- What is the unconditional (i.e. marginal) pdf of **Y**?
- $f_Y(v) = \int_{u=-\infty}^{\infty} f_{X,Y}(u,v)\,du = \int_{u=0}^{1-v} 1/(1-u)\,du$



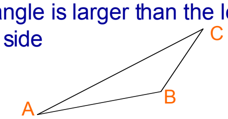
$= -\ln v,\ 0 < v < 1,$ and 0 otherwise

Example F: Popping the question…

- The unit length stick has been broken into three pieces of lengths **X**, **Y**, and 1–**X**–**Y**
- X is uniformly distributed on (0,1)
- Conditioned on **X** = α, **Y** is uniformly distributed on the interval $(0,1-\alpha)$
- The joint pdf is $f_{X,Y}(u,v) = 1/(1-u)$ for $0 < v < 1-u < 1$
- What is the probability that the three pieces can be arranged to form a triangle?

Example F: The triangle inequality

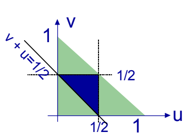
- The triangle inequality is a fundamental result in Euclidean geometry
- The sum of the lengths of any two sides of a triangle is larger than the length of the third side



- length(AB) + length(BC) > length(AC)
- length(AB) + length(AC) > length(BC)
- length(AC) + length(BC) > length(AB)

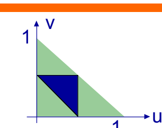
Example F: In terms of X and Y

- The lengths are **X**, **Y**, and 1–**X**–**Y**
- The conditions for a triangle are
  - **X** + **Y** > 1–**X**–**Y**, i.e. **X**+**Y** > 1/2
  - **X** + 1–**X**–**Y** > **Y**, i.e. **Y** < 1/2
  - **Y** + 1–**X**–**Y** > **X**, i.e. **X** < 1/2



Triangle can be formed if (**X**,**Y**) is in the blue region

Is that your final answer?



Triangle can be formed if (**X**,**Y**) is in the blue region

- P{(**X** ,**Y**) ∈ blue region}

$$= \int_{u=0}^{1/2}\int_{v=1/2-u}^{1/2} 1/(1-u)\,dv\,du = \int_{u=0}^{1/2} u/(1-u)\,du$$

$$= \int_{u=0}^{1/2} 1/(1-u) - u\,du = \ln 2 - 1/2 \approx 0.193\ldots$$

Probabilities via conditioning

- For any given region B in the plane, P{(**X**,**Y**) ∈ B} can also be computed by conditioning on the value of **Y** and then using the theorem of total probability

$$P\{(X,Y) \in B\} = \iint_B f_{X,Y}(u,v)\,du\,dv$$

$$P\{(X,Y) \in B\} = \int P(\{(X,Y) \in B\} \mid \alpha) f_Y(\alpha)\,d\alpha$$

Example F: a re-interpretation

- Let T denote the event that a triangle can be formed from the three pieces
- Given that **X** = α where $\alpha < 1/2$,

  P{T|**X** = α}, the conditional probability of being able to form a triangle, is just the the conditional probability that $1/2-\alpha <$ **Y** $< 1/2$
- But, **Y** is conditionally uniform on $(0,1-\alpha)$
- P{T|**X** = α} = $[1/2-(1/2-\alpha)]/(1-\alpha) = \alpha/(1-\alpha)$

Example F: more re-interpretation

- Given that **X** = α where $\alpha < 1/2$

  P{T|**X** = α} = $[1/2-(1/2-\alpha)]/(1-\alpha) = \alpha/(1-\alpha)$
- Given that **X** = α where $\alpha > 1/2$,

  P{T|**X** = α} = 0
- To find P{T}, we multiply this conditional probability by the pdf of **X** and integrate

$$P(T) = \int_{\alpha=0}^{1/2} \alpha/(1-\alpha)\, d\alpha + \int_{\alpha=1/2}^{1} 0\, d\alpha = \ln 2 - 1/2$$

Example F: self-study exercises

- Exercises: Since the triangle cannot be formed if **X** > 1/2, repeat Example F with
  - **X** being uniformly distributed on (0,1/2) instead of (0,1); **Y** is conditionally uniform on $(0,1-\alpha)$ as before
  - **X** being uniformly distributed on (0,1/2) and **Y** having conditional pdf $f_{Y|X}(v|\alpha) = 2\bullet(1-\alpha-v)/(1-\alpha)^2$ for $0 \le v \le 1-\alpha$

    This is the pdf of **Y** in Example E

If conditional probability comes,…

- $f_{Y|X}(v|\alpha)$, the conditional pdf of **Y** given that **X** = α describes the probabilistic behavior of **Y** in the light of the partial knowledge of the outcome of the experiment
- Generally speaking, $f_{Y|X}(v|\alpha)$ is not the same as $f_{Y|X}(v|\beta)$
- But, what if $f_{Y|X}(v|\alpha)$ is the same for all choices of the number α?
- $f_Y(v) = \int_{\alpha=-\infty}^{\infty} f_{Y|X}(v|\alpha) f_X(\alpha)\, d\alpha = f_{Y|X}(v|\alpha)$

…can independence be far behind?

- If $f_{Y|X}(v|\alpha)$ is the same for all choices of the number α, then $f_{Y|X}(v|\alpha)$ is the same as $f_Y(v)$, the unconditional pdf of **Y**
- Knowing the value α that **X** took on tells us nothing more than we knew already: the modified (conditional) pdf is the same as the unconditional pdf
- In such cases, **X** and **Y** are said to be mutually independent random variables

Independent random variables

- Since $f_{Y|X}(v|\alpha) = f_{X,Y}(u,\alpha)/f_X(\alpha)$, the joint pdf $f_{X,Y}(u,v)$ can be written as

  $$f_{X,Y}(u,v) = f_{Y|X}(v|u)\bullet f_X(u)$$
- If $f_{Y|X}(v|u) = f_Y(v)$ for all u, then

  $f_{X,Y}(u,v) = f_X(u)\bullet f_Y(v)$ for all u and v
- Jointly continuous random variables **X** and **Y** are said to be mutually independent random variables if $f_{X,Y}(u,v) = f_X(u)\bullet f_Y(v)$ for all choices of real numbers u and v

Joint pdf must factor everywhere

- Jointly continuous random variables **X** and **Y** are said to be mutually independent random variables if $f_{X,Y}(u,v) = f_X(u) \cdot f_Y(v)$ for all choices of real numbers u and v
- The random variables are independent if the joint pdf factors into the product of the marginal pdfs at every point in the plane
- There is no such thing as "**X** and **Y** are independent at (3,4) but not at (1,2)"

Joint CDF also factors everywhere

- Jointly continuous random variables **X** and **Y** are said to be mutually independent random variables if $f_{X,Y}(u,v) = f_X(u) \cdot f_Y(v)$ for all choices of real numbers u and v
- This implies that $F_{X,Y}(u,v) = F_X(u) \cdot F_Y(v)$ for all choices of real numbers u and v
- The events $\{X \le u\}$ and $\{Y \le v\}$ are mutually independent events for all choices of real numbers u and v

General notion of independent RVs

- Any two (not necessarily continuous) random variables **X** and **Y** are said to be independent random variables if

  $$F_{X,Y}(u,v) = F_X(u) \cdot F_Y(v)$$

  for all choices of u and v
- As noted on the previous slide, **X** and **Y** are independent RVs if the events $\{X \le u\}$ and $\{Y \le v\}$ are mutually independent events for all choices of u and v

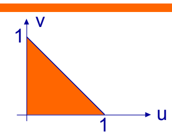
Joint pmf factors too!

- When **X** and **Y** are discrete random variables, factorization of the joint CDF into the product of the marginal CDFs is equivalent to the statement that

  $$p_{X,Y}(u,v) = p_X(u) \cdot p_Y(v)$$

  for all choices of u and v
- $P\{X = u_i, Y = v_j\} = P\{X = u_i\} \cdot P\{Y = v_j\}$
- The events $\{X = u_i\}$ and $\{Y = v_j\}$ are independent for all i and j

The eyeball test



- If a joint pdf is nonzero on the region shown, we can assert that **X** and **Y** are dependent RVs without any calculations
- **Y** can take on values between 0 and 1
- But, if **X** = 1/2, **Y** can take on values between 0 and 1/2 only! DEPENDENT!!

More on the eyeball test

- Joint pdf is nonzero on a region other than a rectangular region (with sides parallel to the axes)?
  X and **Y** are dependent random variables
- Joint pmf matrix of discrete RVs **X** and **Y** has zero entries?
  X and **Y** are dependent random variables
- The eyeball test proves dependence only. If the joint pdf ≠ 0 on a rectangle, you still need to check if $f_{X,Y}(u,v) = f_X(u) f_Y(v)\ \forall\ u,v$

Generalization to n RVs – I

- n random variables are said to be independent if their joint CDF factors into the product of the marginal CDFs
- $F_{\underline{X}}(\underline{u}) = P\{\underline{\mathbf{X}} \le \underline{u}\}$
  $= P\{\mathbf{X}_1 \le u_1, \mathbf{X}_2 \le u_2, \ldots, \mathbf{X}_n \le u_n\}$
  $= P\{\mathbf{X}_1 \le u_1\}P\{\mathbf{X}_2 \le u_2\} \ldots P\{\mathbf{X}_n \le u_n\}$
  $= F_{\mathbf{X}_1}(u_1)\bullet F_{\mathbf{X}_2}(u_2)\bullet \ldots \bullet F_{\mathbf{X}_n}(u_n)$
- $\{\mathbf{X}_1 \le u_1\}, \{\mathbf{X}_2 \le u_2\}, \ldots, \{\mathbf{X}_n \le u_n\}$ are independent events for all choices of $\underline{u}$

Generalization to n RVs – II

- For jointly continuous random variables, joint CDF = product of the marginal CDFs implies joint pdf = product of marginal pdfs
- $f_{\underline{X}}(\underline{u}) = f_{\mathbf{X}_1}(u_1)\bullet f_{\mathbf{X}_2}(u_2)\bullet \ldots \bullet f_{\mathbf{X}_n}(u_n)$
- For discrete random variables, the joint pmf factors into the product of the marginal pmfs
- $p_{\underline{X}}(\underline{u}) = p_{\mathbf{X}_1}(u_1)\bullet p_{\mathbf{X}_2}(u_2)\bullet \ldots \bullet p_{\mathbf{X}_n}(u_n)$

Summary — I

- The notion of the conditional pdf of one random variable given the value of another random variable was introduced
- The conditional pdf of **Y** given **X** = α is the ratio of the joint pdf of **X** and **Y** to the marginal pdf of **X** at α
- $f_{\mathbf{Y}|\mathbf{X}}(v|\alpha) = f_{\mathbf{X},\mathbf{Y}}(u,\alpha)/f_{\mathbf{X}}(\alpha)$
- The unconditional pdf of **Y** can be found from the conditional pdf via an integral version of the theorem of total probability

Summary — II

- Random variables **X** and **Y** are said to be independent if $\{\mathbf{X} \le u\}$ and $\{\mathbf{Y} \le v\}$ are independent events for all choices of real numbers u and v
- $F_{\mathbf{X},\mathbf{Y}}(u,v) = F_{\mathbf{X}}(u)\bullet F_{\mathbf{Y}}(v)$ for all choices of real numbers u and v
- For jointly continuous random variables, $f_{\mathbf{X},\mathbf{Y}}(u,v) = f_{\mathbf{X}}(u)\bullet f_{\mathbf{Y}}(v)$ for all choices of real numbers u and v

Summary — III

- For discrete random variables **X** and **Y,** $F_{\mathbf{X},\mathbf{Y}}(u,v) = F_{\mathbf{X}}(u)\bullet F_{\mathbf{Y}}(v)$ is equivalent to the statement that events $\{\mathbf{X} = u_i\}$ and $\{\mathbf{Y} = v_j\}$ are independent for all i and j
- $p_{\mathbf{X},\mathbf{Y}}(u,v) = p_{\mathbf{X}}(u)\bullet p_{\mathbf{Y}}(v)$ for all choices of real numbers u and v
- The eyeball test is an easy method of determing when RVs are dependent
- These notions generalize naturally to the independence of n random variables