Jointly continuous RVs

- Jointly continuous random variables spread the probability mass (usually with varying density) over a region
- The joint pdf $f_{X,Y}(u,v)$ tells us how dense the probability mass is at the point (u,v)
  - There is no probability mass at any point $P\{(\mathbf{X},\mathbf{Y}) = (u,v)\} = P\{\mathbf{X} = u, \mathbf{Y} = v\} = 0$ for all real numbers u and v
  - $f_{X,Y}(u,v)$ is the density of the probability mass. Units are probability mass per unit area

The joint pdf is mass per unit area

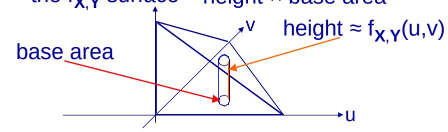
- The joint pdf $f_{X,Y}(u,v)$ is not a probability: we must multiply the pdf by an area to get a probability
- $P\{(\mathbf{X},\mathbf{Y})\ \ \{$small region containing $(u,v)\}\}$ $\approx f_{X,Y}(u,v)\bullet\{$area of region$\}$
- For larger regions, Probability = (double) integral of the pdf over the region
- $P\{(\mathbf{X},\mathbf{Y})\ \ A\} = \int\int_A f_{X,Y}(u,v)\,du\,dv$

Graphical interpretation

- Joint pdf is a surface above the u-v plane
- $P\{(\mathbf{X},\mathbf{Y})\ \ \{$small region containing $(u,v)\}\}$

$\approx f_{X,Y}(u,v)\bullet\{$area of region$\}$

$\approx$ volume above the u-v plane and below the $f_{X,Y}$ surface $\approx$ height $\times$ base area



Probability = volume under joint pdf

$P\{(\mathbf{X},\mathbf{Y})\ \ A\} = \int\int_A f_{X,Y}(u,v)\,du\,dv$

- $P\{u \le \mathbf{X} \le u+\delta u, v \le \mathbf{Y} \le v+\delta v\}$ $\approx f_{X,Y}(u,v)\bullet\delta u\bullet\delta v$ is the volume of a prism of height $f_{X,Y}(u,v)$ and rectangular base of area $\delta u\bullet\delta v$
- $P\{(\mathbf{X},\mathbf{Y})\ \ A\}$ = volume of solid with vertical sides, base A, and varying height $f_{X,Y}(u,v)$ = volume between A and $f_{X,Y}$ surface

Properties of the joint pdf

- $F_{X,Y}(u_0,v_0) = \int_{v=-\infty}^{v_0}\int_{u=-\infty}^{u_0} f_{X,Y}(u,v)\,du\,dv$ or integrate w.r.t. v first and then w.r.t. to u
- $f_{X,Y}(u,v) \ge 0$ for all u and v
- Total volume between the $f_{X,Y}$ surface and the u-v plane is 1
- This is just the interpretation of the result

$F_{X,Y}(\infty,\infty) = 1 = \int_{v=-\infty}^{\infty}\int_{u=-\infty}^{\infty} f_{X,Y}(u,v)\,du\,dv$

Finding joint pdf from the joint CDF

- For jointly continuous RVs,

$f_{X,Y}(u,v) = \frac{\partial^2}{\partial v \partial u} F_{X,Y}(u,v)$ if the derivative exists, and $f_{X,Y}(u,v)$ = any number ≥ 0 otherwise

The set of points where the joint CDF is not differentiable has zero area, e.g. a straight line or curve in the plane

Differentiating an integral

- $\int_{a(\alpha)}^{b(\alpha)} g(x;\alpha)\,dx = G(\alpha)$
- What is the derivative of the integral, that is, the function $G(\alpha)$, with respect to α?
- $\frac{\partial}{\partial\alpha}G(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha}g(x;\alpha)\,dx$
  $+ g(b(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}b(\alpha)$
  $- g(a(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}a(\alpha)$

Special cases

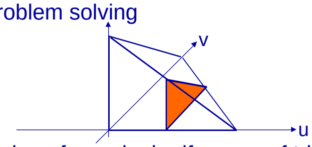
- Special case: The limits do not depend on α; only the integrand depends on α
- $\frac{\partial}{\partial\alpha}G(\alpha) = \int_a^b \frac{\partial}{\partial\alpha}g(x;\alpha)\,dx$
- Special case: Only the limits depend on α; the integrand does not depend on α
- $\frac{\partial}{\partial\alpha}G(\alpha)$
  $= g(b(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}b(\alpha) - g(a(\alpha);\alpha)\bullet\frac{\partial}{\partial\alpha}a(\alpha)$

The marginal pdfs of X and Y

- $f_X(u)$ and $f_Y(v)$, the marginal pdfs of **X** and **Y** respectively, are obtained by "integrating out" the unwanted variable from $f_{X,Y}(u,v)$
- $f_X(u) = \int_{v=-\infty}^{\infty} f_{X,Y}(u,v)\,dv$; $f_Y(v) = \int_{u=-\infty}^{\infty} f_{X,Y}(u,v)\,du$
- $f_X(u) = \int_{v=-\infty}^{\infty} f_{X,Y}(u,v)\,dv$ = area of cross-section of pdf surface by vertical plane through the point u

Think geometrically!

- Thinking of probabilities, pdfs etc in geometrical terms can be a great help in problem solving



- Value of marginal pdf = area of triangle = (1/2)•base•altitude
- Look, Ma! No integration!

Generalization to n RVs – I

- $\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n$ are called jointly continuous random variables if
  - the random point $\underline{\mathbf{X}} = (\mathbf{X}_1, \mathbf{X}_2, \ldots, \mathbf{X}_n)$ takes on all possible values in a region of nonzero volume in n-dimensional space, and
  - The probabilistic behavior is described by the n-variate joint pdf
    $f_{\underline{X}}(\underline{u}) = f_{X_1,X_2,\ldots,X_n}(u_1, u_2, \ldots, u_n)$

Generalization to n RVs – II

- The joint pdf is not a probability: we must multiply by an n-dimensional volume to get the probability
- For small $\underline{\delta u} = (\delta u_1, \delta u_2, \ldots, \delta u_n)$
  $P\{\underline{u} \le \underline{\mathbf{X}} \le \underline{u} + \underline{\delta u}\}$
  = P{$\underline{\mathbf{X}}$ in small rectangular parallelipiped}
  $\approx f_{\underline{X}}(\underline{u})\bullet\delta u_1\bullet\delta u_2\bullet \ldots\bullet\delta u_n$
- For larger volumes, use an n-dimensional integral over the region of interest

Generalization to n RVs – III

- The joint pdf is the first-order partial derivative of the joint CDF with respect to all the variables
- $f_{\underline{X}}(\underline{u}) \geq 0$
- Integral of $f_{\underline{X}}(\underline{u})$ over the n-dimensional space is 1
- n-dimensional generalization of the result

$$F_{X,Y}(\infty,\infty) = 1 = \int_{v=-\infty}^{\infty} \int_{u=-\infty}^{\infty} f_{X,Y}(u,v)\, du\, dv$$

Generalization to n RVs – IV

- The joint pdf of any subset of the random variables is obtained by integrating out the unwanted variables
- This joint pdf is called the marginal pdf of the subset
- If **X**, **Y**, **Z** have joint pdf $f_{X,Y,Z}(u,v,w)$, then

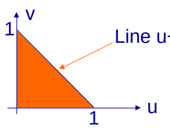
$$f_{X,Z}(u,w) = \int_{v=-\infty}^{\infty} f_{X,Y,Z}(u,v,w)\, dv$$

Example A

- The joint pdf of **X** and **Y** is given by
  $f_{X,Y}(u,v) = c\bullet(1 - u - v)$ for $0 \leq u \leq 1-v \leq 1$
  and $f_{X,Y}(u,v) = 0$ otherwise
  (a) What is the value of c?
  (b) Find the marginal pdf of **X**
- Get DAD* to help: first figure out what the pdf looks like, and draw a sketch showing where the pdf is nonzero
  *DAD = draw a diagram!

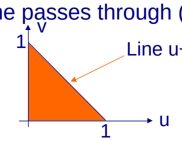
Solution to Example A (pdf region)

- Figuring out the region
- $f_{X,Y}(u,v) = c\bullet(1 - u - v)$ for $0 \leq u \leq 1-v \leq 1$
- What is the region $\{(u,v): 0 \leq u \leq 1-v \leq 1\}$?
- Both u and v must be between 0 and 1
- Furthermore, u + v must also be between 0 and 1



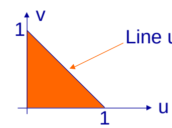
Solution to Example A (pdf shape)

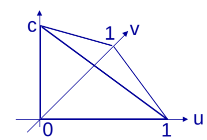
- Figuring out the pdf shape
- $w = c\bullet(1 - u - v)$ is the equation of a plane in 3-dimensional space with axes u,v,w
- The plane passes through the line in the u-v plane with equation u + v = 1
- The plane passes through (0, 0, c)



Solution to Example A (2 diagrams)

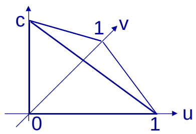
- $f_{X,Y}(u,v) = c\bullet(1 - u - v)$ for $0 \leq u \leq 1-v \leq 1$
  - is nonzero on orange triangular region
  - is a pyramid as shown below





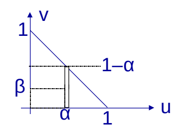
Solution to Example A (finding c)

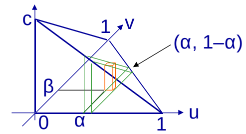
- What is the value of c?
- Total volume under the pdf surface = 1
- Volume of pyramid = base area × height/3 = (1/2×1×1) × c/3 = c/6 c = 6



Solution to Example A (set up ∬)

- For any fixed α between 0 and 1, take an elemental volume at (α,β) as shown
- Vary β from 0 to 1–α to get volume of slice
- Vary α between 0 and 1 to get entire volume of pyramid





Solution to Example A (compute ∬)

- $F_{X,Y}(\infty,\infty) = 1 = \int_{\alpha=-\infty}^{\infty}\int_{\beta=-\infty}^{\infty} f_{X,Y}(\alpha,\beta)\, d\beta\, d\alpha$

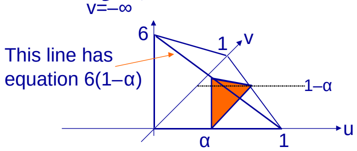
$$= \int_{\alpha=0}^{1}\int_{\beta=0}^{1-\alpha} c\bullet(1-\alpha-\beta)\, d\beta\, d\alpha$$

$$= \int_{\alpha=0}^{1} c\bullet(1-\alpha)^2/2\, d\alpha = c/6$$

- Hence, c = 6, as obtained earlier

Solution to Example A (find $f_X(u)$)

- $f_X(\alpha) = \int_{v=-\infty}^{\infty} f_{X,Y}(\alpha,v)\, dv$ = area of cross-section



- $f_X(\alpha)$ = area of triangle = base•altitude/2 = (1–α)•[6•(1–α)]/2 = 3•(1–α)^2

Solution to Example A (find $f_X(u)$)

- $f_X(u) = \int_{v=-\infty}^{\infty} f_{X,Y}(u,v)\, dv$ = area of cross-section
- $f_X(u) = 3\bullet(1-u)^2$????
- Not quite! Our analysis applies only when u is between 0 and 1
- If u < 0 or u > 1, the cross-section is non-existent and thus has zero area
- $f_X(u) = 3\bullet(1-u)^2$ for $0 \le u \le 1$, and $f_X(u) = 0$ otherwise

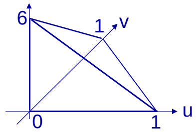
Solution to Example A (crosscheck)

- $f_X(u) = 3\bullet(1-u)^2$ for $0 \le u \le 1$, and $f_X(u) = 0$ otherwise
- An important check on the correctness of the work done is the test that the marginal pdf is indeed a valid pdf
- Test failed? Your calculations are wrong
- Test passed? Your calculations may still be wrong, but they are not obviously incorrect
- Exercise: Verify that $f_X(u)$ is a valid pdf

Solution to Example A (symmetry)

- $f_X(u) = 3\cdot(1-u)^2$ for $0 \le u \le 1$, and $f_X(u) = 0$ otherwise
- By symmetry,

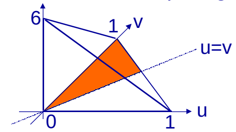
  $f_Y(v) = 3\cdot(1-v)^2$ for $0 \le v \le 1$, and $f_Y(v) = 0$ otherwise



Extension of Example A (symmetry)

- What is P{**X** < **Y**}?
- P{**X** < **Y**} = P{(**X**,**Y**) orange triangle}

  = 1/2 by symmetry
- We could have written this down even if he had made an error in computing c = 6!



Example B

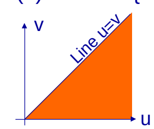
- The joint pdf of **X** and **Y** is given by

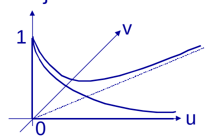
  $f_{X,Y}(u,v) = \exp(-u)$ for $0 \le v \le u < \infty$

  and $f_{X,Y}(u,v) = 0$ otherwise

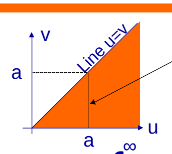
  (a) Find the marginal pdfs of **X** and **Y**

  (b) Find P{**X** + **Y** ≤ α} where α ≥ 0



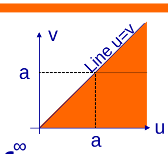


Example B – marginal pdf of X



- $f_X(a) = \int_{v=-\infty}^{\infty} f_{X,Y}(a,v)\,dv$ = area of cross-section
- Cross-section is a rectangle of base a and height exp(–a)
- $f_X(u) = u\cdot\exp(-u),\ u \ge 0$ **X** is Γ(2,1) RV

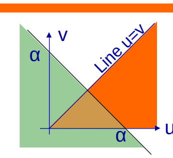
Example B – marginal pdf of Y



- $f_Y(a) = \int_{u=-\infty}^{\infty} f_{X,Y}(u,a)\,du$ = area of cross-section

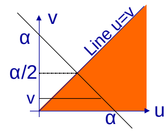
  $= \int_{u=a}^{\infty} \exp(-u)\,du = \exp(-a)$
- $f_Y(v) = \exp(-v),\ v \ge 0$ **Y** = exp. RV

Example B – P{X + Y ≤ α}



- P{**X** + **Y** ≤ α} = integral of $f_{X,Y}(u,v)$ over green shaded region
- Easier if we integrate with respect to u first and then with respect to v

Example B – P{X + Y ≤ α}



- For any fixed value of v, 0 ≤ v ≤ α/2, u varies from v to α–v
- $P\{\mathbf{X} + \mathbf{Y} \le \alpha\} = \int_{v=0}^{\alpha/2} \int_{u=v}^{\alpha-v} \exp(-u)\, du\, dv$

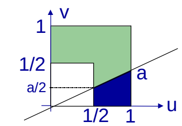
  $= 1-2\bullet\exp(-\alpha/2)+\exp(-\alpha) = (1 - \exp(-\alpha))^2$

Uniform distribution over a region

- The random point (**X**,**Y**) is said to be uniformly distributed over a region A if the joint pdf has constant value over A (and is zero elsewhere)
- $f_{X,Y}(u,v)$ = constant = 1/{area of A}
- For any given region B in the plane P{(**X**,**Y**) B} = {area of A∩B}/{area of A} = $f_{X,Y}(u,v)$•{area of A∩B}
- Generalization to n random variables

Example C — P{Y ≤ aX} = ?

- The random point (**X**,**Y**) is uniformly distributed over the shaded region

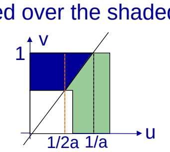


- $f_{X,Y}(u,v)$ = 4/3 on the shaded region
- P{**Y** ≤ a**X**} = (4/3)•area of blue region

  = (4/3)•(3a/8) = a/2. Note: a ≤ 1

Example C — when a ≥ 1

- The random point (**X**,**Y**) is uniformly distributed over the shaded region



- For 1 ≤ a < ∞,

  P{**Y** ≤ a**X**} = 1 – (4/3)•area of blue region

  = 1 – (4/3)•(3/8a) = 1 – 1/2a

Summary

- We reviewed the fundamentals of the theory of two random variables
- We worked through several illustrative examples
- We attempted to emphasize the use of graphical visualization, intuition, and symmetry as problem-solving tools
- We introduced the concept of a uniform distribution over a region

