## ECE 313
### Probability with Engineering Applications
### Jointly Continuous Random Variables I

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### Probability mass on the real line

- The pmf $p_X(u)$ for a discrete random variable **X** describes a collection of point masses on the real line: $P\{X = u\} = p_X(u)$
- A continuous random variable **X** spreads the mass on (an interval of) the real line
  - There is no probability mass at any point $P\{X = u\} = 0$ for all real numbers u
  - pdf $f_X(u)$ is the density of the probability mass Units are probability mass per unit length
  - $P\{X \in$ {small interval that contains number u}} $\approx f_X(u)\bullet$\{length of the interval\}



### Joint behavior of continuous RVs

- When **X** and **Y** are discrete RVs, the random point (**X**,**Y**) is also discrete valued
- The joint pmf $p_{X,Y}(u,v)$ describes point masses in the plane
- If **X** and **Y** are continuous RVs, then either
  - (**X**,**Y**) can take on all possible values in a region of nonzero area

  or
  - **Y** = g(**X**), and thus (**X**,**Y**) always lies on the curve v = g(u) in the plane



### Probability mass in the plane

- Suppose that **X** and **Y** are continuous RVs, and the random point (**X**,**Y**) can take on all values in a region of nonzero area
- In this case, **X** and **Y** are said to be jointly continuous RVs
- The probability mass is spread over this region of the plane
- If **Y** = g(**X**), then the probability mass is spread along the curve v = g(u)
  **X** and **Y** are not jointly continuous RVs



### Probability mass along a curve

- If **Y** = g(**X**), then the probability mass is spread along the curve v = g(u)
- **X** and **Y** are not jointly continuous RVs
- All questions involving the probabilistic behavior of the random point (**X**,**Y**) can be translated into questions involving **X** alone
- Example: If $Y = X^2$, then for any $u,v \geq 0$, the joint CDF of **X** and **Y** is:
  $F_{X,Y}(u,v) = P\{X \leq u, Y \leq v\} = P\{X \leq u, X^2 \leq v\}$
  $= P\{-a \leq X \leq a\}$ where $a = \min\{u, \sqrt{v}\}$

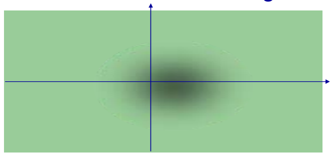


### Probability mass over a region

- The case of **X** and **Y** being continuous random variables but not jointly continuous random variables is easily treated via the methods that we have studied previously
- The interesting case is when **X** and **Y** are jointly continuous and the probability mass is spread over a region of the plane
- Abuse of language: We will assume that, unless explicitly stated otherwise, continuous RVs are also jointly continuous

## Jointly continuous RVs

- **X** and **Y** are jointly continuous RVs
- The random point (**X**,**Y**) can take on all possible values in the oval region shown

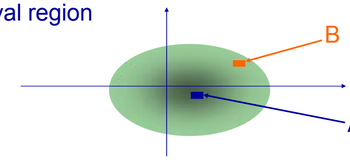


- The region can be the entire plane too



## Probabilities may differ…

- **X** and **Y** are jointly continuous RVs
- The random point (**X**,**Y**) is always in the oval region




- But, P{(**X**,**Y**) ∈ subregion A} may be different from P{(**X**,**Y**) ∈ subregion B}



## Joint probability density function

- The probability mass is spread (usually with varying density) over the region
- The joint probability density function (joint pdf) $f_{X,Y}(u,v)$ tells us how dense the probability mass is at the point (u,v)
  - There is no probability mass at any point P{(**X**,**Y**) = (u,v)} = P{**X** = u, **Y** = v} = 0 for all real numbers u and v
  - $f_{X,Y}(u,v)$ is the density of the probability mass Units are probability mass per unit area

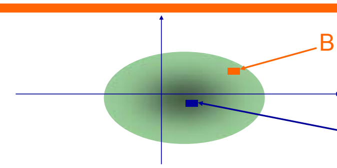


## Generalization of the notion of pdf

- The pdf of a RV is not a probability
- We must multiply the pdf by a length to get a probability: works best for short lengths
- More generally, probability = integral of pdf
- Similarly, the joint pdf $f_{X,Y}(u,v)$ is not a probability: we must multiply the pdf by an area to get a probability
- P{(**X**,**Y**) ∈ {small region containing (u,v)}} ≈ $f_{X,Y}(u,v)$•{area of region}
- Probability = (double) integral of pdf



## Joint pdf has different values…




- Even when regions A and B have the same area, P{(**X**,**Y**) ∈ region A} might be different from P{(**X**,**Y**) ∈ region B}

  because the joint pdf has different value in those regions



## Integral is the limit of a sum…

- P{(**X**,**Y**) ∈ {small region containing (u,v)}} ≈ $f_{X,Y}(u,v)$•{area of region}
- The approximation is good for small areas
- For large regions, subdivide the region into very small subregions and add up these approximations to the probabilities
- In the limit as the number of the small subregions increases, we get the integral of $f_{X,Y}(u,v)$ over the large region

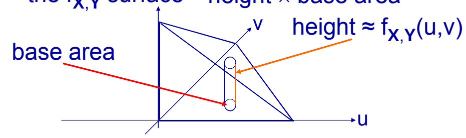
## Probability = integral of joint pdf

- The probability that the random point (**X**,**Y**) will have value in a given region A is the integral of the joint pdf over the region A
- $P\{(\mathbf{X},\mathbf{Y}) \in A\} = \iint_A f_{\mathbf{X},\mathbf{Y}}(u,v)\,du\,dv$
- This is NOT a magic formula
- Before evaluating the double integral, it is necessary to first set up the limits of integration correctly



## Graphical interpretation

- Joint pdf is a surface above the u-v plane
- $P\{(\mathbf{X},\mathbf{Y}) \in \{\text{small region containing } (u,v)\}\}$
  $\approx f_{\mathbf{X},\mathbf{Y}}(u,v) \bullet \{\text{area of region}\}$
  $\approx$ volume above the u-v plane and below the $f_{\mathbf{X},\mathbf{Y}}$ surface $\approx$ height $\times$ base area






## Probability = volume under joint pdf

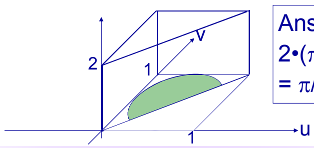
$P\{(\mathbf{X},\mathbf{Y}) \in A\} = \iint_A f_{\mathbf{X},\mathbf{Y}}(u,v)\,du\,dv$

- $P\{u \le \mathbf{X} \le u+\delta u, v \le \mathbf{Y} \le v+\delta v\}$
  $\approx f_{\mathbf{X},\mathbf{Y}}(u,v) \bullet \delta u \bullet \delta v$ is the volume of a prism of height $f_{\mathbf{X},\mathbf{Y}}(u,v)$ and rectangular base of area $\delta u \bullet \delta v$
- $P\{(\mathbf{X},\mathbf{Y}) \in A\}$ = volume of solid with vertical sides, base A, and varying height $f_{\mathbf{X},\mathbf{Y}}(u,v)$
  = volume between A and $f_{\mathbf{X},\mathbf{Y}}$ surface



## Think about finding the volume!

- Probability should be thought of first as a volume — then as an integral if necessary
- Example: $f_{\mathbf{X},\mathbf{Y}}(u,v) = 2$ for $0 < u < v < 1$ and is 0 otherwise.
- What is $P\{(\mathbf{X} - 0.5)^2 + (\mathbf{Y} - 0.5)^2 \le 0.5^2\}$?

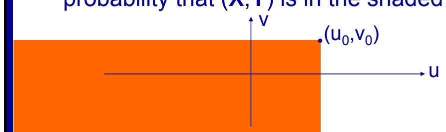



Answer:
$2 \bullet (\pi \bullet 0.5^2)/2$
$= \pi/4$



## The joint CDF is simple to find…

- $F_{\mathbf{X},\mathbf{Y}}(u_0,v_0) = P\{\mathbf{X} \le u_0, \mathbf{Y} \le v_0\}$ is the probability that (**X**,**Y**) is in the shaded area




- $F_{\mathbf{X},\mathbf{Y}}(u_0,v_0) = \int_{v=-\infty}^{v_0} \int_{u=-\infty}^{u_0} f_{\mathbf{X},\mathbf{Y}}(u,v)\,du\,dv$
  or integrate w.r.t. v first and then w.r.t. to u



## Properties of the joint pdf

- The joint pdf has two properties
  - $f_{\mathbf{X},\mathbf{Y}}(u,v) \ge 0$ for all u and v
    Since probability is nonnegative, its density is also nonnegative
  - Total volume between $f_{\mathbf{X},\mathbf{Y}}$ surface and the u-v plane is 1
- This is just the interpretation of the result

$$F_{\mathbf{X},\mathbf{Y}}(\infty,\infty) = 1 = \int_{v=-\infty}^{\infty} \int_{u=-\infty}^{\infty} f_{\mathbf{X},\mathbf{Y}}(u,v)\,du\,dv$$

## Analogous to one-dimensional pdfs

- The pdf of a continuous RV is nonnegative: the total area between the pdf curve and the u-axis is 1
- $F_X(u_0)$ = area under pdf curve to left of $u_0$
- The joint pdf of jointly continuous RVs is also nonnegative: total volume between the joint pdf surface and the u-v plane is 1
- $F_{X,Y}(u_0,v_0)$ = volume under pdf surface in SW quadrant with NE corner $(u_0,v_0)$



## Find joint pdf from the joint CDF

- The pdf of a continuous RV is the derivative of its CDF
- Similarly for jointly continuous RVs,

  $f_{X,Y}(u,v) = \frac{\partial^2}{\partial v \partial u} F_{X,Y}(u,v)$ if the derivative exists

  and $f_{X,Y}(u,v)$ = any number $\geq 0$ otherwise
- The set of points where the joint CDF is not differentiable has zero area, e.g. a straight line or curve in the plane



## Differentiating an integral – I

- To understand why the double partial derivative of $F_{X,Y}(u,v)$ is the joint pdf, we need to understand how to differentiate an integral
- This notion is very handy in working with joint distributions
- Objection: Isn't an integral an area? So, its value is a constant and therefore, the derivative of the integral is 0, right?



## Differentiating an integral – II

- $g(x;\alpha)$ = function of x; $\alpha$ is a parameter
- Example: $g(x;\alpha) = \exp(-\alpha x)$
- $\int_a^b g(x;\alpha)\,dx$ = function of $\alpha$ $= \exp(-a\alpha)-\exp(-b\alpha)$
- The limits might depend on $\alpha$ too, e.g. $a(\alpha) = \alpha$ and $b(\alpha) = \alpha^2$
- $\int_{a(\alpha)}^{b(\alpha)} g(x;\alpha)\,dx = G(\alpha) = \exp(-\alpha^2) - \exp(-\alpha^3)$



## Differentiating an integral – III

- $\int_{a(\alpha)}^{b(\alpha)} g(x;\alpha)\,dx = G(\alpha)$
- What is the derivative of the integral, that is, the function $G(\alpha)$, with respect to $\alpha$?
- $\frac{\partial}{\partial\alpha}G(\alpha) = \int_{a(\alpha)}^{b(\alpha)} \frac{\partial}{\partial\alpha} g(x;\alpha)\,dx + g(b(\alpha);\alpha)\cdot\frac{\partial}{\partial\alpha}b(\alpha) - g(a(\alpha);\alpha)\cdot\frac{\partial}{\partial\alpha}a(\alpha)$



## Differentiating an integral – IV

- Special case: The limits do not depend on $\alpha$; only the integrand depends on $\alpha$
- $\frac{\partial}{\partial\alpha}G(\alpha) = \int_a^b \frac{\partial}{\partial\alpha} g(x;\alpha)\,dx$
- Special case: Only the limits depend on $\alpha$; the integrand does not depend on $\alpha$
- $\frac{\partial}{\partial\alpha}G(\alpha) = g(b(\alpha);\alpha)\cdot\frac{\partial}{\partial\alpha}b(\alpha) - g(a(\alpha);\alpha)\cdot\frac{\partial}{\partial\alpha}a(\alpha)$

## From the CDF to the pdf

- $F_{X,Y}(u,v) = \int_{x=-\infty}^{u} \int_{y=-\infty}^{v} f_{X,Y}(x,y)\, dy\, dx$
- Evaluate the inner integral first
- The result is a function of x and v that we denote by g(x; v)
- $F_{X,Y}(u,v) = \int_{x=-\infty}^{u} g(x;v)\, dx$
- $\frac{\partial}{\partial u} F_{X,Y}(u,v) = g(u;v) = \int_{y=-\infty}^{v} f_{X,Y}(u,y)\, dy$



## From the CDF to the pdf – finish up

- $F_{X,Y}(u,v) = \int_{x=-\infty}^{u} \int_{y=-\infty}^{v} f_{X,Y}(x,y)\, dy\, dx$
- $\frac{\partial}{\partial u} F_{X,Y}(u,v) = g(u;v) = \int_{y=-\infty}^{v} f_{X,Y}(u,y)\, dy$
- Take the partial derivative with respect to v
- $\frac{\partial}{\partial v}\frac{\partial}{\partial u} F_{X,Y}(u,v) = \frac{\partial^2}{\partial v \partial u} F_{X,Y}(u,v) = f_{X,Y}(u,v)$
- Note that it does not matter in which order we take the partial derivatives



## The marginal pdfs of X and Y

- The joint continuity of **X** and **Y** implies that **X** and **Y** are themselves continuous RVs
- The pdfs $f_X(u)$ and $f_Y(v)$ can be obtained from the joint pdf $f_{X,Y}(u, v)$
- $f_X(u)$ and $f_Y(v)$ are called the marginal pdfs of **X** and **Y** in the sense of "being obtained from the joint pdf"
- As is the case with CDFs and pmfs, the word marginal is not pejorative



## Obtaining the marginal pdfs

- $F_X(u) = F_{X,Y}(u,\infty) = \int_{x=-\infty}^{u} \int_{y=-\infty}^{\infty} f_{X,Y}(x,y)\, dy\, dx$
- $f_X(u)$, the pdf of **X** is just the derivative of its CDF $F_X(u)$
- Denote the inner integral by g(x)
- $\frac{\partial}{\partial u} F_X(u) = g(u) = \int_{y=-\infty}^{\infty} f_{X,Y}(u,y)\, dy$
- $f_X(u) = \int_{v=-\infty}^{\infty} f_{X,Y}(u,v)\, dv$; $f_Y(v) = \int_{u=-\infty}^{\infty} f_{X,Y}(u,v)\, du$



## Integrate out the unwanted variable

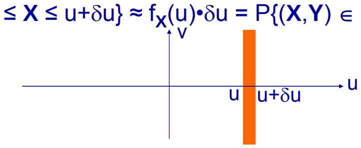
- $f_X(u)$ and $f_Y(v)$, the marginal pdfs of **X** and **Y** respectively, are obtained by "integrating out" the unwanted variable from $f_{X,Y}(u,v)$
- $f_X(u) = \int_{v=-\infty}^{\infty} f_{X,Y}(u,v)\, dv$; $f_Y(v) = \int_{u=-\infty}^{\infty} f_{X,Y}(u,v)\, du$
- Compare: for discrete RVS, the marginal pmfs are obtained by summing over the unwanted variables in the joint pmf
- Analogous result: Integral = limit of a sum



## Graphical interpretations

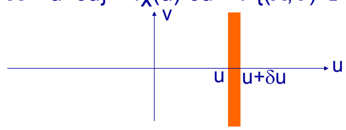
$F_X(u) = P\{X \le u\} = P\{(X,Y) \in \text{region shown}\}$


$P\{u \le X \le u+\delta u\} \approx f_X(u) \cdot \delta u = P\{(X,Y) \in \text{strip}\}$

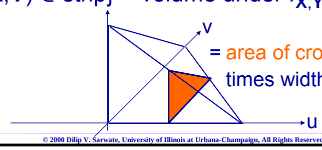


## Just a sliver please, I'm on a diet…

$P\{u \le \mathbf{X} \le u+\delta u\} \approx f_X(u)\bullet\delta u = P\{(\mathbf{X},\mathbf{Y}) \in \text{strip}\}$




$P\{(\mathbf{X},\mathbf{Y}) \in \text{strip}\}$ = volume under $f_{X,Y}$ surface




= area of cross-section times width $\delta u$ of strip

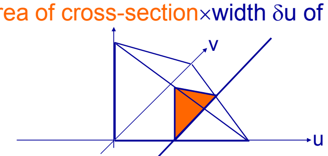


## Fix u, and vary v as we integrate…

$P\{u \le \mathbf{X} \le u+\delta u\} \approx f_X(u)\bullet\delta u = P\{(\mathbf{X},\mathbf{Y}) \in \text{strip}\}$

$P\{(\mathbf{X},\mathbf{Y}) \in \text{strip}\}$ = volume under $f_{X,Y}$ surface

= area of cross-section × width $\delta u$ of strip




- $f_X(u) = \int_{v=-\infty}^{\infty} f_{X,Y}(u,v)\,dv$ = area of cross-section



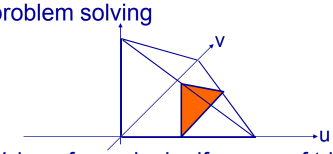
## Interpretation of marginal pdf value

- $f_X(u) = \int_{v=-\infty}^{\infty} f_{X,Y}(u,v)\,dv$ = area of cross-section
- u is a fixed number, say u = 5.13
- As we vary v in the integral, we are integrating the function $f_{X,Y}(5.13,v)$
- $f_X(5.13)$ = Area of the cross-section of the $f_{X,Y}(u,v)$ surface when it is sliced by a plane parallel to the v axis through the point u = 5.13



## Think geometrically!

- Thinking of probabilities, pdfs etc in geometrical terms can be a great help in problem solving




- Value of marginal pdf = area of triangle
  = (1/2)•base•altitude
- Look, Ma! No integration!



## Summary

- We have studied the probabilistic behavior of jointly continuous random variables
- The joint pdf describes the density of the probability mass at each point in the plane
- For a region of small area containing (u,v),

$P\{(\mathbf{X},\mathbf{Y}) \in \text{region}\} \approx f_{X,Y}(u,v)\bullet\{\text{area of region}\}$

- For larger areas, $P\{(\mathbf{X},\mathbf{Y}) \in \text{region}\}$ is obtained by integrating the joint pdf over the region



## Summary (continued)

- The integral of the joint pdf over a region is the volume between the $f_{X,Y}(u,v)$ surface and the region in the u-v plane
- The value of the joint CDF at point $(u_0,v_0)$ is the integral of the joint pdf $f_{X,Y}(u,v)$ over the SW quadrant with NE corner at $(u_0,v_0)$ = volume under joint pdf in the quadrant
- The joint pdf $f_{X,Y}(u,v)$ is the double partial derivative of the joint CDF $F_{X,Y}(u,v)$ with respect to u and v

## Summary (continued some more)

- The marginal pdfs $f_X(u)$ and $f_Y(v)$ are obtained by "integrating out" the unwanted variable from the joint pdf $f_{X,Y}(u,v)$
- The value of the marginal pdf for any number u is the area of the cross-section of the joint pdf at the point u
- Thinking of probabilities, pdfs, etc in geometrical terms as volumes, areas, etc is a great help in problem solving



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