Life and Death

- Previous studies of system reliability
  - Very simple model
  - System is working or has failed
  - P{system is working} = p
  - P{system has failed} = q = 1–p
- We study system reliability once again
- What is the probability that a system works for T time units (seconds/hours/days)?
- What is the average lifetime of a system?

Lifetime (death time?) of a system

- A system is put into operation at time t = 0
- Assumption: the system actually is in working condition at t = 0
- At some later time, the system fails
- The time of failure cannot be predicted, and is modeled as a random variable **X**
- **X** denotes the age of the system at death
- **X** is the length of life for the system = lifetime of the system = time of death

X is a positive random variable

- Since the system is working at t = 0, its lifetime **X** can take on positive values only
- Model **X** as a continuous RV with pdf f(u)
- f(u) = 0 for u ≤ 0
- E[**X**] is the average lifetime of the system
- E[**X**] is also called the
  - mean time to failure (MTTF)
  - mean time before failure (MTBF)

How does MTBF make any sense?

- A system begins to operate at t = 0
- As soon as it fails, it is replaced by a brand new system
- When the substitute fails, it is replaced by yet another new system, … and so on
- The experiment consists of measuring (or observing) the lifetime of a system
- Successive trials of the experiment provide independent observations of **X**

Long-term average ≈ expectation

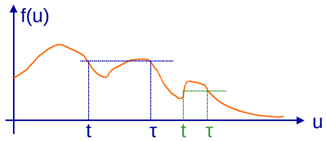
- If N systems together provided service for a total of T hours under the "replace the system when it fails" policy, the observed long-term average of the N lifetimes is T/N
- Long-term average T/N ≈ expectation E[**X**]
- MTBF is defined to be E[**X**]
- If service has to be provided for T hours using systems whose MTBF is L, then, on average, expect to need T/L systems

What does f(u) look like?

- f(u) = 0 for u ≤ 0
- F(u) = 0 for u ≤ 0
- If f(u) = 0 for all u > T, the system is guaranteed to fail by time T !!
- F(T) = P{**X** ≤ T} = P{system lifetime ≤ T}
  = area under f(u) up to T = 1
- In most of the interesting cases, it is useful to assume that f(u) > 0 for all u > 0
- Note that f(u) decays away to 0 as u → ∞

Simple notions from limits

- $f(u) > 0$ for all $u > 0$
- $f(u)$ decays away to 0 as $u \to \infty$



- Given any number $t > 0$, we can find a $\tau \geq t$ such that $f(u) < f(t)$ for all $u \geq \tau$

An awkward fact …

- Given any $t > 0$, we can find a $\tau \geq t$ such that $f(u) < f(t)$ for all $u \geq \tau$
- For small values of ε
  - $P\{t < \mathbf{X} \leq t + \varepsilon\} \approx f(t)\cdot\varepsilon$
  - $P\{\tau < \mathbf{X} \leq \tau + \varepsilon\} \approx f(\tau)\cdot\varepsilon$

$P\{t < \mathbf{X} \leq t+\varepsilon\} \approx f(t)\cdot\varepsilon > f(\tau)\cdot\varepsilon \approx P\{\tau < \mathbf{X} \leq \tau + \varepsilon\}$

- The chances of dying within the next 24 hours (= ε) are larger at age $t = 21$ years than at age $\tau = 95$ years??

Shouldn't you be scared to death?

- The chances of dying within the next 24 hours (= ε) are larger at age $t = 21$ years than at age $\tau = 95$ years !!
- Why is an older system less likely to fail than a newer one?
- Answer: "You should live so long!"
- The chances of dying within the next 24 hours at age 95 are of interest mostly to those who have already lived to be 95!!

A more interesting probability

- P{dying within the next 24 hours} is of interest only to those alive right now
- $P\{t < \mathbf{X} \leq t + \varepsilon\}$ is not of as much interest as the conditional probability $P[\{t < \mathbf{X} \leq t + \varepsilon\} \mid \{\mathbf{X} > t\}]$

$$= \frac{P[\{t < \mathbf{X} \leq t + \varepsilon\} \cap \{\mathbf{X} > t\}]}{P\{\mathbf{X} > t\}}$$

$$= \frac{P\{t < \mathbf{X} \leq t + \varepsilon\}}{P\{\mathbf{X} > t\}} \approx \frac{f(t)\cdot\varepsilon}{1 - F(t)}$$

The hazard rate function

- $P[\{t < \mathbf{X} \leq t + \varepsilon\} \mid \{\mathbf{X} > t\}] = f(t)\cdot\varepsilon/[1 - F(t)]$ is the (conditional) probability that a system that is working at time t fails within the next ε time units
- $P[\{t < \mathbf{X} \leq t + \varepsilon\} \mid \{\mathbf{X} > t\}] \approx h(t)\cdot\varepsilon$ where

$$h(t) = f(t)/[1 - F(t)]$$

is called the hazard rate function of the system (or of the random variable $\mathbf{X}$)
- hazard = chance of danger or risk

Also called the failure rate function

$$h(t) = f(t)/[1 - F(t)]$$

is also called the failure rate function of the system (or of the random variable $\mathbf{X}$)

- $h(21)\cdot(1/365) \approx$ chances of dying within the next 24 hours if you are 21 years old
- $h(95)\cdot(1/365) \approx$ chances of dying within the next 24 hours if you are 95 years old
- In most cases, $h(95) >> h(21)$ and

$$h(95)\cdot(1/365) >> h(21)\cdot(1/365)$$

What the hazard rate function is not

- Many people hold various wrong beliefs
  - ~~h(t) is the pdf of the lifetime **X**~~
    - We shall see a little later that the area under the h(t) curve is infinite, so that h(t) cannot be a pdf (nor can c•h(t) for any c!)
  - ~~h(t) is the conditional pdf of **X**~~
  - ~~h(t) is the probability of failure at time t~~
    - We shall see a little later that h(t) can exceed 1, so it cannot be a probability

So, what is h(t) anyway?

- h(t) tells you that the probability that a system (that happens to be working at time t) fails before time t+ε is ≈ h(t)•ε
- P{system is still working at time t + ε} ≈ 1 – h(t)•ε Note: remember ε is small
- $h(t_2) > h(t_1)$ means a system in working condition at time t_2 is more likely to fail "real soon now" than a system in working condition at time t_1

Finding h(t) from the pdf of X

- We have assumed that f(u) > 0 for all u > 0
- Hence, F(t) < 1 for all t because there is always area under the pdf to the right of t
- Given f(u), find the CDF F(u)
- Then, h(t) = f(t)/[1 – F(t)] for all t > 0
- We are not dividing by 0 in this calculation
- If f(u) = 0 for all u > x, then h(t) → ∞ as t → x, and h(t) = ∞ for all t ≥ x

Examples

- Example: **X** is an exponential RV with parameter λ
- f(u) = λ•exp(–λu), and F(t) = 1 – exp(–λt)
- Hence, h(t) = λ = constant!
- Example: **X** has a Rayleigh pdf given by $f(u) = (u/\sigma^2)\bullet\exp[-u^2/(2\sigma^2)]$
- $F(t) = 1 - \exp[-t^2/(2\sigma^2)]$
- Hence, $h(t) = t/\sigma^2$: hazard rate increases linearly with time (i.e. age of the system)

A digression on Rayleigh RVs

- Rayleigh RVs arise as models for lifetimes in system reliability studies
- They model the fading of RF signals as they propagate through the ionosphere
- They model the amplitude of thermal noise at the output of narrowband filters
- pdf has a peak of (1/σ)•exp(–1/2) at u = σ
- $E[\mathbf{X}] = \sigma\bullet(\pi/2)^{1/2}$
- $E[\mathbf{X}^2] = 2\sigma^2$

Look, Ma! No integrations!

- $E[\mathbf{X}] = \int u\bullet f(u)du = \int (u/\sigma)^2\bullet\exp[-u^2/(2\sigma^2)]du$ from 0 to ∞
- Set $t = u^2/(2\sigma^2)$ to get $E[\mathbf{X}] = \Gamma(1/2)\bullet\sigma/\sqrt{2}$
- Alternatively, E[**X**] is the integral of 1 – F(t) $= \exp[-t^2/(2\sigma^2)]$ from 0 to ∞
- But we know that this integral is $\sigma(2\pi)^{1/2}/2$
- Why?
- Hence, $E[\mathbf{X}] = \sigma\bullet(\pi/2)^{1/2}$

Finding the pdf from h(t)

- Isn't this backwards? Isn't the pdf usually given? and we know already how to find h(t) from f(u)
- Hazard rates can be measured easily
- Turn on 100,000 light bulbs
- Each morning, check what fraction of those that were burning bright yesterday are still lit today. Fraction = h(t)

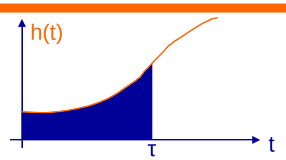
Finding the pdf from h(t)

- Given the pdf f(u), we can find the hazard rate h(t) = f(t)/[1 – F(t)] for all t > 0
- Given h(t), what is f(u)?
- The area under the curve h(t) from t = 0 to t = τ must be the same as the area under the curve f(t)/[1 – F(t)]
- Integrate to find the area under the curve
- Remember limits!!

Integrate, and don't forget limits!

- $\int_0^{\tau} h(t)\,dt = \int_0^{\tau} \frac{f(t)}{1 - F(t)}\,dt = -\ln(1 - F(t))\Big|_0^{\tau}$
  $= -\ln(1 - F(\tau))$ since $F(0) = 0$
- Hence,
  $F(\tau) = 1 - \exp\left(-\int_0^{\tau} h(t)\,dt\right)$ for $\tau > 0$
- For any positive real number τ, the value of the CDF of **X** at τ, ***viz.*** F(τ), is given by 1 – exp (– area under h(t) to the left of τ)

A graphical interpretation



- For any positive real number τ, the value of the CDF of **X** at τ, ***viz.*** F(τ), is given by 1 – exp (– area under h(t) to the left of τ) = 1 – exp(– blue area)

What happens at ∞ ?

- $F(\infty) = 1$
- F(∞) = 1 – exp (– total area under h(t))
- Hence, the total area under the hazard rate curve h(t) must be infinite
- $\int_0^{\infty} h(t)\,dt = \infty$
- h(t) cannot be a pdf of anything
- c•h(t) cannot be a pdf for any choice of c

Area under hazard rate curve = ∞

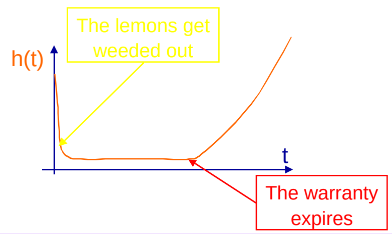
- Area under the hazard rate curve is infinite
- As a general rule, h(t) → ∞ as t → ∞, so this is easily satisfied
- For an exponential random variable with parameter λ, h(t) = λ = constant
- Is this a realistic model?
- Yes! Semiconductor devices do seem to exhibit this type of behavior

Do ICs never fail?

- For semiconductor devices, the hazard rate appears to be a constant
- Does this mean that ICs never fail?
- No, it means that ICs don't seem to age
- A 20-year-old IC is just as likely to fail within the next hour as a brand new one!
- Get real! How many 20-year-old ICs are still in service?

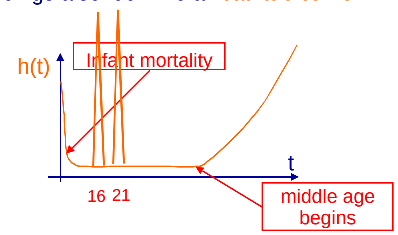
More typical hazard rate functions

- Typical hazard rate functions look like the "bathtub curve" shown below



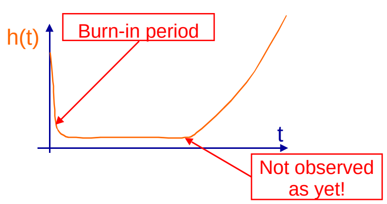
For human beings too!

- Typical hazard rate functions for human beings also look like a "bathtub curve"



For semiconductor devices too!

- Hazard rate functions for semiconductor devices also look like a "bathtub curve"



Live long and prosper

- The lifetimes $\mathbf{X}_1$ and $\mathbf{X}_2$ of two different systems have hazard rates $h_1(t)$ and $h_2(t)$ respectively
- Which system has the longer average lifetime? Which is larger: $E[\mathbf{X}_1]$ or $E[\mathbf{X}_2]$?
- $E[\mathbf{X}] = \int_0^\infty [1 - F(\tau)]\, d\tau = \int_0^\infty P\{\mathbf{X} > \tau\}\, d\tau$
- $1 - F(\tau) = \exp\left(-\int_0^\tau h(t)\, dt\right)$ for $\tau > 0$

Greater hazard rate shorter life

- $E[\mathbf{X}_i] = \int_0^\infty [1 - F_i(\tau)]\, d\tau = \int_0^\infty P\{\mathbf{X}_i > \tau\}\, d\tau$
- $1 - F_i(\tau) = \exp\left(-\int_0^\tau h_i(t)\, dt\right)$ for $\tau > 0$
- Suppose $h_1(t) > h_2(t)$ for all $t > 0$. Then,
  $\int h_1(t)\, dt > \int h_2(t)\, dt \quad 1 - F_1(\tau) < 1 - F_2(\tau)$
- $1 - F_1(\tau) < 1 - F_2(\tau) \quad E[\mathbf{X}_1] < E[\mathbf{X}_2]$
- $h_1(t) > h_2(t) \quad E[\mathbf{X}_1] < E[\mathbf{X}_2]$

The conditional CDF of X

- We noted earlier that the hazard rate was not the conditional pdf of the lifetime **X**
- So, what is a conditional pdf anyway?
- Analogous to conditional pmfs
- Given that event A (with P(A) > 0) has occurred, the conditional CDF of **X** is

$$F_{X|A}(u|A) = P\{X \le u|A\}$$

- As usual, $P\{X \le u|A\} = P(\{X \le u\} \cap A)/P(A)$ denotes a conditional probability

Conditional pdf = derivative

- Given A has occurred, the conditional CDF of **X** is $F_{X|A}(u|A) = P\{X \le u|A\}$
- Conditional CDFs enjoy all the properties of ordinary CDFs, i.e. they are monotone nondecreasing, right-continuous functions with limits 0 at –∞ and 1 at ∞
- Under the same conditions as regular CDFs, the derivative of the conditional CDF is the conditional pdf $f_{X|A}(u|A)$

The conditional pdf of X

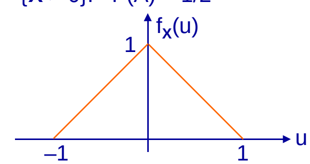
- The derivative of the conditional CDF $F_{X|A}(u|A)$ is the conditional pdf $f_{X|A}(u|A)$
- Conditional pdfs have the same properties as regular pdfs
  - $f_{X|A}(u|A) \ge 0$ for all u
  - Total area under curve = 1
  - $f_{X|A}(u|A) \cdot \Delta u \approx P\{u \le X \le u + \Delta u\ |A\}$ for small values of Δu

Conditional mean and variance of X

- We can calculate means and variances using conditional pdfs
  - The mean of the conditional pdf is called the conditional mean $E[X|A]$
  - The variance of the conditional pdf is called the conditional variance $\mathrm{var}(X|A)$
- We are merely using the conditional pdf $f_{X|A}(u|A)$ instead of the unconditional pdf $f_X(u)$ in the various integrals

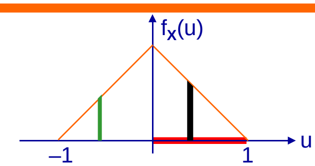
Example

- Let **X** have pdf $f_X(u) = 1 - |u|$ for $|u| < 1$ and 0 otherwise
- Let A = {**X** > 0}. P(A) = 1/2



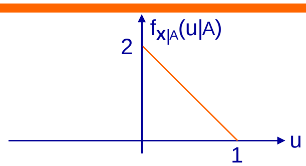
- What is $f_{X|A}(u|A)$?

Example (continued)



- A = {**X** > 0}. P(A) = 1/2
- For –1 < u < 0, $P\{u \le X \le u + \Delta u\ |A\} = 0$
- For 0 < u < 1, $P\{u \le X \le u + \Delta u\ |A\} = P\{u \le X \le u + \Delta u\}/P(A) \approx 2 \cdot f_X(u) \cdot \Delta u$

Example (conclusion)



- A = {**X** > 0}. P(A) = 1/2
- $f_X(u) = 1 - |u|$ for $|u| < 1$; 0 otherwise
- $f_{X|A}(u|A) = 2\bullet(1-|u|) = 2\bullet(1-u)$ for $0 < u < 1$; and 0 otherwise

General conclusion

- If event A has been specified in terms of the random variable **X**, i.e. as an interval or union of intervals on the real line, then the conditional pdf of **X** given A is given by
  - $f_{X|A}(u|A) = f_X(u)/P(A)$ if u A
  - $f_{X|A}(u|A) = 0$ if u A^c
- P(A) = area under $f_X(u)$ in region A is a "normalizing factor" that makes $f_{X|A}(u|A)$ a valid density with area 1

Summary

- The hazard rate of a positive random variable is given by $h(t) = f(t)/[1 - F(t)]$
- $h(t)\bullet\varepsilon \approx P[\{t < \mathbf{X} \le t + \varepsilon\} \,|\, \{\mathbf{X} > t\}]$
- $F(\tau) = 1 - \exp\left(-\int_0^\tau h(t)\,dt\right)$ for $\tau > 0$
- Greater hazard rates correspond to shorter average lifetimes
- We began studying conditional pdfs for continuous random variables