Continuous random variables

- Definition: A continuous random variable **X** is one whose CDF $F_X(u)$ is
  - continuous at all u, $-\infty < u < \infty$
  - differentiable at all u (except possibly at a set of points $u_1 < u_2 < \ldots u_n < \ldots$)
- More precisely, any finite-length interval contains at most finitely many points where $F_X(u)$ is not differentiable

The derivative of the CDF

- The derivative of the CDF of a continuous random variable **X** exists for almost all real numbers u
- Definition: The probability density function (pdf) of a continuous random variable **X** is

$$f_X(u) = \begin{cases} \frac{d}{du} F_X(u), & \text{if } F_X(u) \text{ is differentiable} \\ \text{any number} \geq 0 & \text{if CDF is non-diff.} \end{cases}$$

What values can X take on?

- For every real number u, $P\{\mathbf{X} = u\} = 0$
- The set of all possible values that **X** can take on is indicated by defining $f_X(u)$ explicitly (by means of some formula, say) at those values
- The set of all possible values that **X** cannot take on is indicated by the word elsewhere in the definition of the pdf, as in $f_X(u) = 0$ elsewhere

The pdf versus the pmf

- A discrete random variable defines a set of point masses on the axis: total mass = 1
- In contrast, a continuous random variable defines a spread of the total probability mass of 1 along the axis
- There is no mass at any point
- The pdf of a continuous random variable tells the density of the mass at each point

So, what's a pdf mean, anyway?

- The probability density function (pdf) of a continuous random variable tells us the density of the probability mass at each point on the axis
- pdf is measured in units of mass/length
- The concept of the pdf is analogous to the more common concepts of the mass density, charge density, etc

pdfs are not what you think they are

- The probability density function (pdf) of a continuous random variable is not, by itself, a probability
- Example: $f_X(u) = 3u^2$ for $0 \leq u \leq 1$, and 0 elsewhere
- This pdf has value 3 at u = 1 and 3/4 at 0.5
- This means the probability mass is four times as dense at u = 1 as at u = 0.5

Multiply by a length to get a mass

- The probability density function (pdf) of a continuous random variable is not, by itself, a probability
- If $f_X(u)$ is positive at the point a, then
  $P\{a \le \mathbf{X} \le a + \delta a\} \approx f_X(a)\bullet\delta a$
- Note that δa is the length of the interval
- The approximation is better and better as δa becomes smaller and smaller

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Why does this work?

$P\{a \le \mathbf{X} \le a + \delta a\} \approx f_X(a)\bullet\delta a$

- $f_X(a)$ = derivative of the CDF at point a
  $= \text{limit}_{\Delta a \to 0} [F_X(a+\Delta a) - F_X(a)]/\Delta a$
- But this means that for small values of δa,
  $P\{a \le \mathbf{X} \le a + \delta a\} = F_X(a+ \delta a) - F_X(a)$
  $\approx f_X(a)\bullet\delta a$
- Furthermore, the smaller the value of δa, the better the approximation

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More generally, …

- If $f_X(u)$ is positive at the point u, then
  $P\{u \le \mathbf{X} \le u + \delta u\} \approx f_X(u)\bullet\delta u$
- Let $n\bullet\delta a = b - a$. Then, $P\{a < \mathbf{X} < b\}$
  $= P\{a \le \mathbf{X} \le a+\delta a\} + P\{a+\delta a \le \mathbf{X} \le a+2\bullet\delta a\}$
  $+ \ldots + P\{a+(n-1)\bullet\delta a \le \mathbf{X} \le a+n\bullet\delta a\}$
- $P\{a < \mathbf{X} < b\} \approx f_X(a)\bullet\delta a + f_X(a+ \delta a)\bullet\delta a + \ldots$
  = area under $f_X(u)$ from a to b
  in the limit as δa goes to 0

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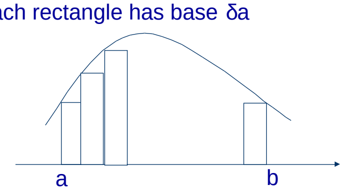
Graphical interpretation

- $P\{a < \mathbf{X} < b\} \approx f_X(a)\bullet\delta a + f_X(a+ \delta a)\bullet\delta a + \ldots$
  = area under $f_X(u)$ from a to b
  in the limit as δa goes to 0
- Each rectangle has base δa



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Probability = area under the curve

- $P\{a < \mathbf{X} < b\} = P\{a \le \mathbf{X} \le b\} = P\{a \le \mathbf{X} < b\}$ $= P\{a < \mathbf{X} \le b\}$ are all equal to
  the area under the curve $f_X(u)$ from a to b
- The area can be expressed as an integral, but it is of the utmost importance that you remember this notion as

  area under the curve

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Think area first, not integral

- $P\{a < \mathbf{X} < b\}=$ area under the curve $f_X(u)$ from a to b
- Calculus defines the area under a curve as the value of the integral
- Think of probability as an area first, and maybe, as an integral later (if necessary)

$$P\{a < \mathbf{X} < b\} = \int_a^b f_X(u)\, du$$

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Definite integrals, please…

- Probabilities are areas
- To find probabilities, we have to use definite integrals, and not antiderivatives (which are also called indefinite integrals)
- We do not need to use indefinite integrals in ECE 313

$$P\{a < \mathbf{X} < b\} = \int_a^b f_X(u)\,du$$

Genius versus stupidity

- “The difference between genius and stupidity is that there are limits to genius”
- Be a genius at ECE 313
- Make sure that your integrals have limits
- Even better yet, make sure that the limits are constants, and not just u as shown

$\int_0^u f_X(u)\,du$ makes no sense whatsoever

Getting help from DAD…

- Most problems on continuous random variables are very much easier to do when the pdf is sketched in a diagram, and the probability to be computed is marked as an area on this diagram
- The diagram will help you in figuring out the limits and help you avoid blunders
- Draw a diagram, for crying out loud!

An example

- Example: A continuous random variable **X** has pdf $f(u) = 0.75(1–u^2)$ for $–1 \le u \le 1$, and 0 otherwise
- $P\{0.25 \le \mathbf{X} \le 1.25\} = \int_{0.25}^{1.25} 0.75(1–u^2)\,du$

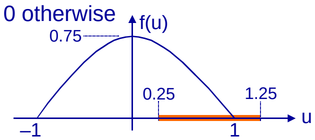
$$= 0.75(u–u^3/3)\Big|_{0.25}^{1.25}$$

$$= 17/64$$

0

DAD helps us with the example

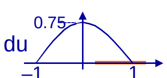
- $f(u) = 0.75(1–u^2)$ for $–1 \le u \le 1$, and 0 otherwise



- $P\{0.25 \le \mathbf{X} \le 1.25\}$ =area under pdf curve from 0.25 to 1.25
  =area under pdf curve from 0.25 and 1 !!

Example (done right)

- Example: A continuous random variable **X** has pdf $f(u) = 0.75(1–u^2)$ for $–1 \le u \le 1$, and 0 otherwise
- $P\{0.25 \le \mathbf{X} \le 1.25\} = \int_{0.25}^{1.25} f(u)\,du$



$$= \int_{0.25}^{1} 0.75(1–u^2)\,du = 0.75(u–u^3/3)\Big|_{0.25}^{1}$$

$$= 81/256$$

- Couldn’t I have done that without DAD?

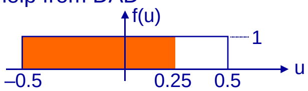
Another example

- Example: A continuous random variable **X** has pdf f(u) = 1 for $-0.5 \le u \le 0.5$, and 0 otherwise
- P{**X** ≤ 0.25} = $\int 1\, du = u$
- To find P{**X** ≤ 0.25}, substitute u = 0.25 in the result of the above integration to get P{**X** ≤ 0.25} = 0.25
- P{**X** ≤ 0.25} = $\int_0^{0.25} 1\, du = 0.25$

0

Example (continued)

- Example: A continuous random variable **X** has pdf f(u) = 1 for $-0.5 \le u \le 0.5$, and 0 otherwise
- Get help from DAD



- P{**X** ≤ 0.25} = area under pdf to left of 0.25 = shaded area = 0.75

Special case

- P{$-\infty$ < **X** < ∞}=the area under the curve $f_X(u)$ from $-\infty$ to ∞
  = total area under the pdf
- But, obviously, P{$-\infty$ < **X** < ∞} = P(Ω) = 1
- Moral: total area under the pdf curve $f_X(u)$ from $-\infty$ to ∞ is 1

  Fancy statement: $\int_{-\infty}^{\infty} f_X(u)\, du = 1$

Defining properties of pdfs

- Any function f(u) that satisfies the two properties
  - f(u) ≥ 0 for all u, $-\infty < u < \infty$
  - $\int_{-\infty}^{\infty} f(u)\, du = 1$

  is a valid pdf in the sense that there is some continuous random variable whose pdf happens to be f(u)

More generally, …

- If a function f(u) is such that
  - f(u) ≥ 0 for all u, $-\infty < u < \infty$
  - $\int_{-\infty}^{\infty} f(u)\, du = c$ where c > 0 is finite

  then, f(u)/c is a valid pdf in the sense that there is some continuous random variable whose pdf happens to be f(u)/c
- This also works if f(u) ≤ 0 for all u
- Why?

A system of checks and balances

- The pdf is a nonnegative function that has unit area (between the pdf curve and the horizontal axis)
- These properties provide a very useful test in checking computations
- If you are asked to find a pdf, and come up with a function that is negative, or whose area is not equal to one, check your work!

I've seen my fondest hope decay…

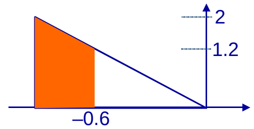
- The pdf is a nonnegative function that has unit area between the pdf curve and the horizontal axis
- As the argument u tends to ±∞, the pdf curve must decay away to 0
- $f_X(+\infty) = \lim_{u \to +\infty} f_X(u) = 0$
- $f_X(-\infty) = \lim_{u \to -\infty} f_X(u) = 0$
- Compare to $F_X(+\infty) = 1$ and $F_X(-\infty) = 0$

Relationship between CDF and pdf

- The pdf is the derivative of the CDF
- $F_X(5) = P\{\mathbf{X} \le 5\} = P\{-\infty < \mathbf{X} \le 5\}$
  = area under pdf $f_X(u)$ from $-\infty$ to 5
- The CDF is not the antiderivative (or indefinite integral) of the pdf
- Ignore the integral formula for the CDF given on page 185 of Ross
- Work it out from first principles

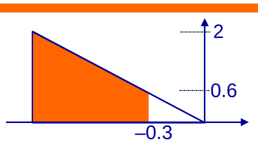
Finding the CDF from the pdf

- Example: $f(u) = -2u$ for $-1 \le u \le 0$, and 0 otherwise
- $F(-0.6) = P\{\mathbf{X} \le -0.6\} = P\{-\infty < \mathbf{X} \le -0.6\}$
  = area under pdf from $-\infty$ to -0.6



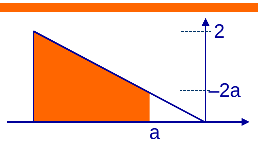
- $F(-0.6) = P\{\mathbf{X} \le -0.6\} = 1 - (1/2)\cdot 0.6 \cdot 1.2$
  $= 1 - (0.6)^2 = 0.64$

Other values of the CDF



- $F(-0.3) = P\{\mathbf{X} \le -0.3\} = P\{-\infty < \mathbf{X} \le -0.3\}$
  = area under pdf from $-\infty$ to -0.3
- $F(-0.3) = P\{\mathbf{X} \le -0.3\} = 1 - (1/2)\cdot 0.3 \cdot 0.6$
  $= 1 - (0.3)^2 = 0.91$
- Hey, Ma! I think I see a pattern here!

More generally, for $-1 \le a \le 0$, …



- For any number a where $-1 \le a \le 0$,
  $F(a) = P\{\mathbf{X} \le a\} = 1 - (1/2)\cdot|a|\cdot|-2a|$
  $= 1 - a^2$
- Gimme's: $F(a) = 0$ for any $a < -1$
  $F(a) = 1$ for any $a > 0$

ECE 313 Survival Guide

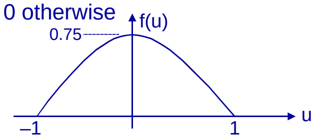
- When dealing with continuous random variables, sketch the pdf (or CDF) curve before you do anything else
- State the desired probability in terms of an area under the pdf, and indicate this area (e.g. by shading) on the sketch
- If you use an integral to find the area, set the limits with the help of the sketch

ECE 313 Survival Guide (Appendix)

- When dealing with continuous random variables, do not use indefinite integrals
- All integrals must have limits
- The upper limit is $+\infty$ or lower limit is $-\infty$ only in rare cases
- F(u) is not the integral of f(u) from $-\infty$ to ∞
- Do not rely on mystical magical formulas (or the kindness of strangers…)

DAD helps with example, again!

- $f(u) = 0.75(1-u^2)$ for $-1 \le u \le 1$, and 0 otherwise

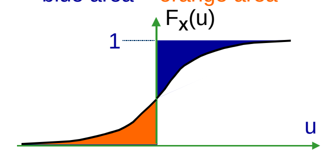


- $P\{\mathbf{X} \le 0\}$ = area under pdf curve to the left of 0 = 1/2 by symmetry!
- Look, Ma! No integration needed!

Expectation of an arbitrary RV

- The expected value $E[\mathbf{X}]$ of an arbitrary random variable $\mathbf{X}$ can be defined as

$$E[\mathbf{X}] = \int_0^\infty [1 - F_X(u)]\,du - \int_{-\infty}^0 F_X(u)\,du$$
= blue area – orange area

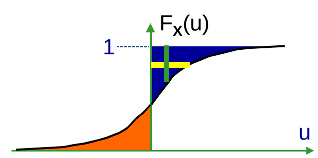


E[X] for continuous RV

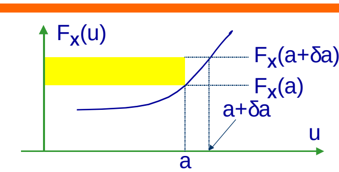
- A Riemann integral is a glorified sum: it is the limit of a Riemann sum
- The integral formula for $E[\mathbf{X}]$ computes the areas in terms of vertical strips
- The same area can be computed by using horizontal strips instead

Expectation via horizontal strips

$$E[\mathbf{X}] = \int_0^\infty [1 - F_X(u)]\,du - \int_{-\infty}^0 F_X(u)\,du$$
= blue area – orange area



An expanded view



- Area of horizontal strip $= a \cdot [F_X(a+\delta a) - F_X(a)]$ $\approx a \cdot f_X(a) \cdot \delta a$ since derivative of CDF = pdf
- Total area = integral of $a \cdot f_X(a)$!!

Expectation of continuous RV

- $E[\mathbf{X}] = \int_{-\infty}^{\infty} u \cdot f_X(u)\, du$
  is a much more useful formula for computational purposes than
  $E[\mathbf{X}] = \int_{0}^{\infty} [1 - F_X(u)]\, du - \int_{-\infty}^{0} F_X(u)\, du$
- Since an integral is a glorified sum, we see that the expectation integral is the analogue of the result $E[\mathbf{X}] = \sum u_i \cdot p(u_i)$

Expectation of continuous RV

- $E[\mathbf{X}] = \int_{-\infty}^{\infty} u \cdot f_X(u)\, du$
  is analogous to $E[\mathbf{X}] = \sum u_i \cdot p(u_i)$ for discrete random variables
- The interpretation in terms of long-term average observed value of **X** also applies
- If we repeat the experiment N times, add up all observed values of **X**, and divide by N, the result will be pretty close to E[**X**]

More on expectation

- $E[\mathbf{X}] = \int_{-\infty}^{\infty} u \cdot f_X(u)\, du$
- All the results regarding expectation that we observed earlier still hold
- LOTUS: If **Y** = g(**X**) where **X** is a continuous random variable with pdf $f_X(u)$, and g(•) is a measurable function, then
  $E[\mathbf{Y}] = E[g(\mathbf{X})] = \int_{-\infty}^{\infty} g(u) \cdot f_X(u)\, du$

Summary

- The probabilistic behavior of a continuous random variable is summarized in its pdf
- The pdf is the derivative of the CDF
- Probabilities are the areas under the pdf
- Sketching the pdf is helpful in computing probabilities
- $E[\mathbf{X}] = \int_{-\infty}^{\infty} u \cdot f_X(u)\, du$; and LOTUS works