Discrete Random Variables

- Recall the generic description of a discrete random variable **X**
- **X** takes on values $u_1, u_2, \ldots u_n, \ldots$ with probabilities $p(u_1), p(u_2), \ldots, p(u_n), \ldots$
- The number of different values is either finite (the list of values terminates at u_n for some n > 1) or is countably infinite
- The values are discretely spaced:

$$u_1 < u_2 < \ldots < u_n < \ldots$$

Cumulative Distribution Function

- The cumulative (probability) distribution function or CDF of a random variable **X** is denoted by F(u) or $F_{\mathbf{X}}(u)$
- Definition: $F_{\mathbf{X}}(u) = P\{\mathbf{X} \le u\}, \ -\infty < u < \infty$
- The CDF is a real-valued function defined for all real number values of its argument u
- Cumulative because the value of the CDF at u is the total probability mass from $-\infty$ up to (and including) the point u

Pejorative remarks on notation

- Most textbook authors (including Ross) use x as the argument of the CDF of a random variable **X**
- They write $P\{\mathbf{X} \le x\}$ instead of $P\{\mathbf{X} \le u\}$ and $F_{\mathbf{X}}(x)$ instead of $F_{\mathbf{X}}(u)$
- This notation causes many problems in students' minds
- What's the difference between **X** and x?
- Which is the RV and which the number?

Doctors and engineers alike…

- The notation $P\{\mathbf{X} \le x\}$ instead of $P\{\mathbf{X} \le u\}$ and $F_{\mathbf{X}}(x)$ instead of $F_{\mathbf{X}}(u)$ causes many problems in students' handwriting
- As the semester wears on (or even right away for the calligraphically challenged), the distinction between **X** and x is lost, as **X** decreases and x increases in size
- All your formulas will look like $P\{x \le x\}$ by the end of the semester and you won't know what it means!

Say it loud and clear…

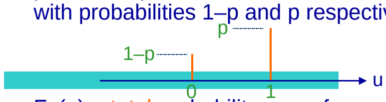
- The notation $P\{\mathbf{X} \le x\}$ instead of $P\{\mathbf{X} \le u\}$ and $F_{\mathbf{X}}(x)$ instead of $F_{\mathbf{X}}(u)$ causes many problems in ordinary speech
- The notation is hard to read aloud
  - $P\{\mathbf{X} \le x\}$ is "probability that big ecks is less than or equal to small ecks"?
  - $F_{\mathbf{X}}(x)$ is read as "eff sub ecks of ecks"?
  - or should it be read as "eff sub big ecks of little ecks"?

Keeping things straight…

- In these days of global-search-and-replace it is not hard to change all occurrences of (x) to (u) and x} to u} etc
- In these slides, random variables are always CAPITALIZED and in **bold face**
- **X** is a random variable — we cannot be sure what its value is going to be
- u is a real variable, a number whose value we can choose as we see fit

Example

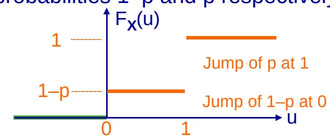
- **X** is a Bernoulli random variable with parameter p. It takes on values 0 and 1 with probabilities 1–p and p respectively



- $F_X(u)$ = total probability mass from –∞ up to (and including) the point u
- $F_X(u) = P\{\mathbf{X} \le u\} = 0$ for any $u < 0$
- $F_X(u) = P\{\mathbf{X} \le u\} = 1–p$ for any u, $0 \le u < 1$
- $F_X(u) = P\{\mathbf{X} \le u\} = 1$ for any $u \ge 1$

Graph of the CDF

- **X** is a Bernoulli random variable with parameter p. It takes on values 0 and 1 with probabilities 1–p and p respectively



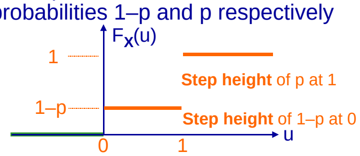
- $F_X(u) = P\{\mathbf{X} \le u\} = 0$ for any $u < 0$
- $F_X(u) = P\{\mathbf{X} \le u\} = 1–p$ for any u, $0 \le u < 1$
- $F_X(u) = P\{\mathbf{X} \le u\} = 1$ for any $u \ge 1$

CDF of a discrete random variable

- The CDF of any discrete random variable is a staircase function
- If **X** takes on values $u_1, u_2, \ldots u_n, \ldots$ with probabilities $p(u_1), p(u_2), \ldots, p(u_n), \ldots$ then the CDF has
  - risers whose step heights are $p(u_1), p(u_2), \ldots, p(u_n), \ldots$
  - at locations $u_1, u_2, \ldots u_n, \ldots$ respectively

… with architectural applications?

- **X** is a Bernoulli random variable with parameter p. It takes on values 0 and 1 with probabilities 1–p and p respectively



- $F_X(u) = P\{\mathbf{X} \le u\} = 0$ for any $u < 0$
- $F_X(u) = P\{\mathbf{X} \le u\} = 1–p$ for any u, $0 \le u < 1$
- $F_X(u) = P\{\mathbf{X} \le u\} = 1$ for any $u \ge 1$

CDF of a discrete random variable

- The CDF of any discrete random variable is a staircase function
- We have defined the u_i to be an increasing sequence $u_1 < u_2 < \ldots < u_n < \ldots$
- For all u in the range $u_i \le u < u_{i+1}$, $F_X(u)$ has value

$$\sum_{j=1}^{i} p(u_j)$$

- $F_X(u)$ jumps in value by $p(u_{i+1})$ at $u = u_{i+1}$

Negative thoughts about the CDF

- The CDF of **X** is far more cumbersome than the pmf
- It took us three lines to write down the “formula” for the CDF of a Bernoulli random variable with parameter p
- The CDF contains no information that we could not have deduced from the pmf
- So, why bother with the CDF at all?

The power of positive thinking

- The concept of the CDF of **X** is a very general one that applies to every random variable, discrete or continuous or mixed…
- Every random variable **X** has a CDF that is defined in exactly the same way:

  $F_X(u) = P\{X \le u\}, \ -\infty < u < \infty$
- But, discrete random variables have pmfs, continuous ones have pdfs, etc …

More power of positive thinking

- Common properties of random variables, i.e. those shared by all random variables, can be stated in terms of the CDF instead of having a different formula for each type of random variable
- The CDF is a unifying concept for all kinds of random variables

A more balanced viewpoint

- Using the general formulations of random variable properties via the CDF is often a messy job — use the pmf or pdf instead
- Using the CDF as an intermediate step is very helpful in some types of calculations and often simplifies these calculations
- The CDF should be used sparingly and for stating general concepts only

So, what's a general CDF look like?

- For any random variable **X**,

  $F_X(u) = P\{X \le u\}, \ -\infty < u < \infty$
- Since the value of the CDF is a probability, we can immediately assert that for all real numbers u,

  $0 \le F_X(u) \le 1$
- The graph of $F_X(u)$ lies between the horizontal or u axis and the line at height 1

CDF is a nondecreasing function

- For any random variable **X**,

  $F_X(u) = P\{X \le u\}, \ -\infty < u < \infty$
- Let a < b. Then, the event $\{X \le a\}$ is a subset of the event $\{X \le b\}$

- $P\{X \le a\} \le P\{X \le b\}$, that is, $F_X(a) \le F_X(b)$

  If a < b, then $F_X(a) \le F_X(b)$

CDF is a nondecreasing function

- For any random variable **X**,

  $F_X(u) = P\{X \le u\}, \ -\infty < u < \infty$
- If a < b, then $F_X(a) \le F_X(b)$
- The value of $F_X(u)$ cannot decrease as u increases from $-\infty$ to $+\infty$
- If $F_X(a) < F_X(b)$, then we can be sure that a < b, but if a < b, then $F_X(a) < F_X(b)$ may not always be true (equality might hold)

CDF increases from 0 to 1

- For any random variable **X**,

  $F_X(u) = P\{\mathbf{X} \le u\}, \quad -\infty < u < \infty$
- The CDF is nondecreasing and bounded between 0 and 1
- $F_X(u)$ approaches limit $F_X(-\infty)$ as $u \to -\infty$
- $F_X(u)$ approaches limit $F_X(+\infty)$ as $u \to +\infty$
- $F_X(-\infty) = 0$ and $F_X(+\infty) = 1$
- **X** takes on values $\pm\infty$ with 0 probability

Can the CDF have jumps?

- The CDF is a nondecreasing function
- The CDF can be a discontinuous function
- Example: The staircase-like CDF of a discrete random variable is discontinuous at each of the possible values of the random variable
- $F_X(u)$ jumps in value by $p(u_i)$ at $u = u_i$ and then remains constant for $u_i \le u < u_{i+1}$, and then jumps again at $u = u_{i+1}$, etc

Limits from left and right

- At a jump discontinuity at $u = t$ (say) in the CDF, the value of $F_X(u)$ just to the left of t is less than the value of $F_X(u)$ just to the right of t
- $F_X(t^-)$ = value of $F_X(u)$ just to the left of t

  = limit of $F_X(u)$ as $u \to t$ from left
- $F_X(t^+)$= value of $F_X(u)$ just to the right of t

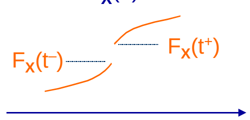
  = limit of $F_X(u)$ as $u \to t$ from right
- $F_X(t^-) < F_X(t^+)$

CDF is a right-continuous function

- $F_X(t^-)$ = value of $F_X(u)$ just to the left of t

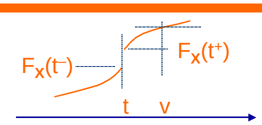
  = limit of $F_X(u)$ as $u \to t$ from left
- $F_X(t^+)$ = value of $F_X(u)$ just to the right of t

  = limit of $F_X(u)$ as $u \to t$ from right



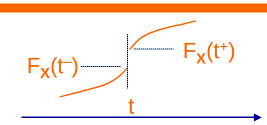
- $F_X(t)$ equals $F_X(t^+)$, the limiting value of $F_X(u)$ as $u \to t$ from right

CDF is a right-continuous function



- At any point t where $F_X(u)$ is discontinuous $F_X(t)$ equals $F_X(t^+)$, the limiting value of $F_X(u)$ as $u \to t$ from right
- At any point of v where $F_X(u)$ is continuous $F_X(v) = F_X(v^+) = F_X(v^-)$

What's the discontinuity?



- At any point t where $F_X(u)$ is discontinuous $F_X(t^+) - F_X(t^-)$= jump in the value of $F_X(u)$

  $= P\{\mathbf{X} = t\}$
- At any point of v where $F_X(u)$ is continuous $F_X(v) = F_X(v^+) = F_X(v^-)$ $\quad P\{\mathbf{X} = v\} = 0$

Probabilities from the CDF

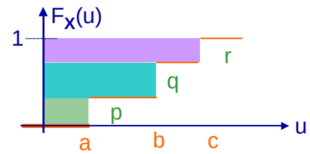
- By definition, $F_X(u) = P\{\mathbf{X} \le u\}$
- $P\{\mathbf{X} = u\} = F_X(u^+) - F_X(u^-)$
  $= $ height of jump (if any) at u
- $P\{\mathbf{X} > u\} = 1 - F_X(u)$
- $P\{\mathbf{X} \ge u\} = 1 - F_X(u^-)$
  $= 1 - F_X(u)$ if CDF is continuous
- $P\{\mathbf{X} < u\} = F_X(u^-)$
  $= F_X(u)$ if CDF is continuous

More probabilities from the CDF

- By definition, $F_X(u) = P\{\mathbf{X} \le u\}$
- $P\{a < \mathbf{X} \le b\} = F_X(b) - F_X(a)$
- $P\{a \le \mathbf{X} \le b\} = F_X(b) - F_X(a^-)$
- $P\{a \le \mathbf{X} < b\} = F_X(b^-) - F_X(a^-)$
- $P\{a < \mathbf{X} < b\} = F_X(b^-) - F_X(a)$
- As in previous cases, the distinctions disappear at points of continuity of the CDF

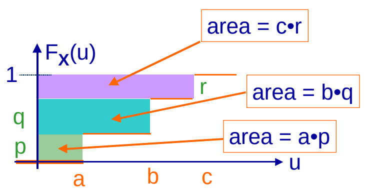
Expectation of positive discrete RV

- Consider a discrete random variable **X** taking on positive values a, b, c with probabilities p, q, r respectively; p+q+r = 1
- $E[\mathbf{X}] = a{\bullet}p + b{\bullet}q + c{\bullet}r$



Expectation of positive discrete RV

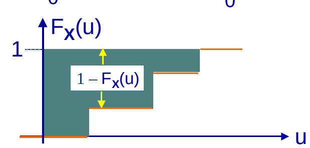
- $E[\mathbf{X}] = a{\bullet}p + b{\bullet}q + c{\bullet}r$
  = area above the CDF and below the line at height 1



Expectation of positive discrete RV

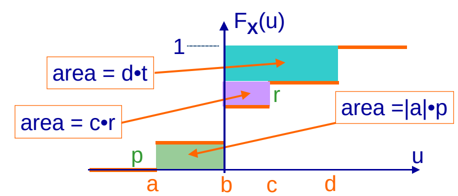
- $E[\mathbf{X}]$ = area above the CDF and below the line at height 1

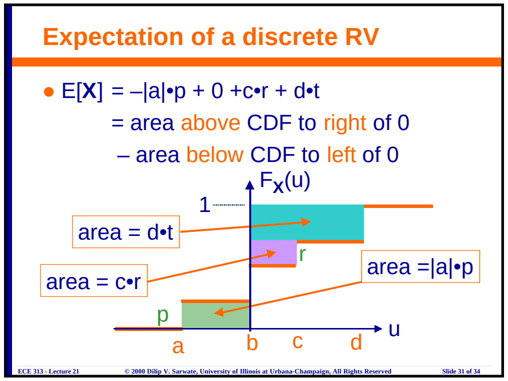
$$= \int_0^{\infty} [1 - F_X(u)]\,du = \int_0^{\infty} P\{\mathbf{X} > u\}\,du$$

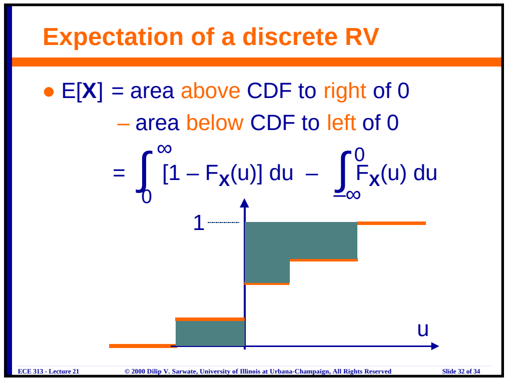


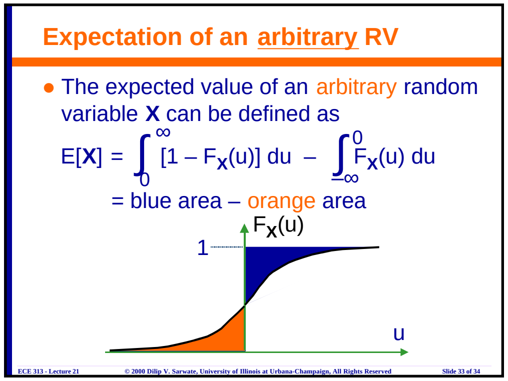
Expectation of a discrete RV

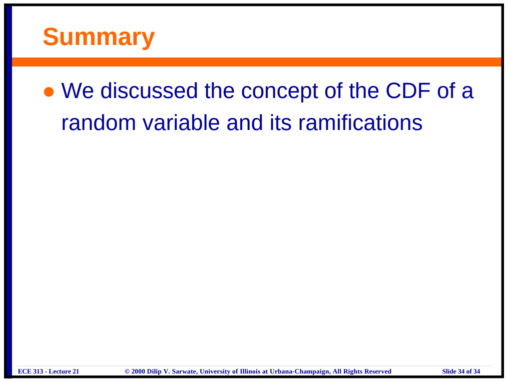
- $E[\mathbf{X}] = a{\bullet}p + b{\bullet}q + c{\bullet}r + d{\bullet}t$
  $= -|a|{\bullet}p + 0 + c{\bullet}r + d{\bullet}t$




Expectation of a discrete RV
• E[X] = –|a|•p + 0 +c•r + d•t
= area above CDF to right of 0
– area below CDF to left of 0
Fx(u)
1
area = d•t
area = c•r
area =|a|•p
r
p
u
a
b
c
d
ECE 313 - Lecture 21
© 2000 Dilip V. Sarwate, University of Illinois at Urbana-Champaign, All Rights Reserved
Slide 31 of 34


Expectation of a discrete RV
• E[X] = area above CDF to right of 0
– area below CDF to left of 0
= ∫₀^∞ [1 – Fx(u)] du – ∫₋∞^0 Fx(u) du
1
u
ECE 313 - Lecture 21
© 2000 Dilip V. Sarwate, University of Illinois at Urbana-Champaign, All Rights Reserved
Slide 32 of 34


Expectation of an arbitrary RV
• The expected value of an arbitrary random variable X can be defined as
E[X] = ∫₀^∞ [1 – Fx(u)] du – ∫₋∞^0 Fx(u) du
= blue area – orange area
Fx(u)
1
u
ECE 313 - Lecture 21
© 2000 Dilip V. Sarwate, University of Illinois at Urbana-Champaign, All Rights Reserved
Slide 33 of 34


Summary
• We discussed the concept of the CDF of a random variable and its ramifications
ECE 313 - Lecture 21
© 2000 Dilip V. Sarwate, University of Illinois at Urbana-Champaign, All Rights Reserved
Slide 34 of 34