Discrete Random Variables

- Recall the generic description of a discrete random variable $X$
- $X$ takes on values $u_1, u_2, \ldots, u_n, \ldots$ with probabilities $p(u_1), p(u_2), \ldots, p(u_n), \ldots$
- The number of different values is either finite (the list of values terminates at $u_n$ for some $n > 1$) or is countably infinite
- The values are discretely spaced: $u_1 < u_2 < \ldots < u_n < \ldots$

Cumulative Distribution Function

- The cumulative (probability) distribution function or CDF of a random variable $X$ is denoted by $F(u)$ or $F_X(u)$
- Definition: $F_X(u) = P\{X \leq u\}$, $-\infty < u < \infty$
- The CDF is a real-valued function defined for all real number values of its argument $u$
- Cumulative because the value of the CDF at $u$ is the total probability mass from $-\infty$ up to (and including) the point $u$

Pejorative remarks on notation

- Most textbook authors (including Ross) use $x$ as the argument of the CDF of a random variable $X$
- They write $P\{X \leq x\}$ instead of $P\{X \leq u\}$ and $F_X(x)$ instead of $F_X(u)$
- This notation causes many problems in students' handwriting
- As the semester wears on (or even right away for the calligraphically challenged), the distinction between $X$ and $x$ is lost, as $X$ decreases and $x$ increases in size
- All your formulas will look like $P\{x \leq x\}$ by the end of the semester and you won’t know what it means!

Doctors and engineers alike...

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Say it loud and clear...

- The notation $P\{X \leq x\}$ instead of $P\{X \leq u\}$ and $F_X(x)$ instead of $F_X(u)$ causes many problems in ordinary speech
- The notation is hard to read aloud
  - $P\{X \leq x\}$ is “probability that big ecks is less than or equal to small ecks”?
  - $F_X(x)$ is read as “eff sub ecks of ecks”?
  - or should it be read as “eff sub big ecks of little ecks”?

Keeping things straight...

- In these days of global-search-and-replace it is not hard to change all occurrences of $(x)$ to $(u)$ and $x)$ to $u)$ etc
- In these slides, random variables are always CAPITALIZED and in bold face
- $X$ is a random variable — we cannot be sure what its value is going to be
- $u$ is a real variable, a number whose value we can choose as we see fit
Example

- $X$ is a Bernoulli random variable with parameter $p$. It takes on values 0 and 1 with probabilities $1-p$ and $p$ respectively.
- $F_X(u) = \text{total probability mass from } -\infty \text{ up to (and including) the point } u$
- $F_X(u) = P\{X \leq u\} = 0$ for any $u < 0$
- $F_X(u) = P\{X \leq u\} = 1-p$ for any $0 \leq u < 1$
- $F_X(u) = P\{X \leq u\} = 1$ for any $u \geq 1$

Graph of the CDF

- $X$ is a Bernoulli random variable with parameter $p$. It takes on values 0 and 1 with probabilities $1-p$ and $p$ respectively.
- $F_X(u) = P\{X \leq u\} = 1-p$ for any $0 \leq u < 1$
- $F_X(u) = P\{X \leq u\} = 1$ for any $u \geq 1$
- $F_X(u)$ jumps in value by $p(u_{i+1})$ at $u = u_{i+1}$

CDF of a discrete random variable

- The CDF of any discrete random variable is a staircase function.
- If $X$ takes on values $u_1, u_2, \ldots u_n$ with probabilities $p(u_1), p(u_2), \ldots, p(u_n)$, then the CDF has:
  - risers whose step heights are $p(u_1), p(u_2), \ldots, p(u_n)$
  - at locations $u_1, u_2, \ldots u_n$ respectively

Negative thoughts about the CDF

- The CDF of $X$ is far more cumbersome than the pmf
- It took us three lines to write down the "formula" for the CDF of a Bernoulli random variable with parameter $p$
- The CDF contains no information that we could not have deduced from the pmf
- So, why bother with the CDF at all?
The power of positive thinking

- The concept of the CDF of $X$ is a very general one that applies to every random variable, discrete or continuous or mixed...
- Every random variable $X$ has a CDF that is defined in exactly the same way:
  \[ F_X(u) = P(X \leq u), \quad -\infty < u < \infty \]
- But, discrete random variables have pmfs, continuous ones have pdfs, etc ...

More power of positive thinking

- Common properties of random variables, i.e. those shared by all random variables, can be stated in terms of the CDF instead of having a different formula for each type of random variable
- The CDF is a unifying concept for all kinds of random variables

A more balanced viewpoint

- Using the general formulations of random variable properties via the CDF is often a messy job — use the pmf or pdf instead
- Using the CDF as an intermediate step is very helpful in some types of calculations and often simplifies these calculations
- The CDF should be used sparingly and for stating general concepts only

So, what’s a general CDF look like?

- For any random variable $X$, 
  \[ F_X(u) = P(X \leq u), \quad -\infty < u < \infty \]
- Since the value of the CDF is a probability, we can immediately assert that for all real numbers $u$,
  \[ 0 \leq F_X(u) \leq 1 \]
- The graph of $F_X(u)$ lies between the horizontal or $u$ axis and the line at height 1

CDF is a nondecreasing function

- For any random variable $X$, 
  \[ F_X(u) = P(X \leq u), \quad -\infty < u < \infty \]
- Let $a < b$. Then, the event \( \{X \leq a\} \) is a subset of the event \( \{X \leq b\} \)
  \[ P(X \leq a) \leq P(X \leq b), \text{ that is, } F_X(a) \leq F_X(b) \]
  If $a < b$, then $F_X(a) \leq F_X(b)$

CDF is a nondecreasing function

- For any random variable $X$, 
  \[ F_X(u) = P(X \leq u), \quad -\infty < u < \infty \]
- If $a < b$, then $F_X(a) \leq F_X(b)$
- The value of $F_X(u)$ cannot decrease as $u$ increases from $-\infty$ to $+\infty$
- If $F_X(a) < F_X(b)$, then we can be sure that $a < b$, but if $a < b$, then $F_X(a) < F_X(b)$ may not always be true (equality might hold)
CDF increases from 0 to 1

- For any random variable $X$, $F_X(u) = P(X \leq u)$, $-\infty < u < \infty$
- The CDF is nondecreasing and bounded between 0 and 1
- $F_X(u)$ approaches limit $F_X(-\infty)$ as $u \to -\infty$
- $F_X(u)$ approaches limit $F_X(+\infty)$ as $u \to +\infty$
- $F_X(-\infty) = 0$ and $F_X(+\infty) = 1$
- $X$ takes on values $\pm \infty$ with 0 probability

Can the CDF have jumps?

- The CDF is a nondecreasing function
- The CDF can be a discontinuous function
- Example: The staircase-like CDF of a discrete random variable is discontinuous at each of the possible values of the random variable
- $F_X(u)$ jumps in value by $p(u_i)$ at $u = u_i$ and then remains constant for $u_i \leq u < u_{i+1}$, and then jumps again at $u = u_{i+1}$, etc

Limits from left and right

- At a jump discontinuity at $u = t$ (say) in the CDF, the value of $F_X(u)$ just to the left of $t$ is less than the value of $F_X(u)$ just to the right of $t$
- $F_X(t^-) = \text{value of } F_X(u) \text{ just to the left of } t$
- $F_X(t^+)= \text{value of } F_X(u) \text{ just to the right of } t$
- $F_X(t^-)$ is the limiting value of $F_X(u)$ as $u \to t$ from left
- $F_X(t^+)$ is the limiting value of $F_X(u)$ as $u \to t$ from right

CDF is a right-continuous function

- $F_X(t^-) = \text{value of } F_X(u) \text{ just to the left of } t$
- $F_X(t^+)= \text{value of } F_X(u) \text{ just to the right of } t$
- $F_X(t^-) = \text{limit of } F_X(u) \text{ as } u \to t$ from left
- $F_X(t^+)$ is the limiting value of $F_X(u)$ as $u \to t$ from right
- $F_X(t^-) = F_X(t^+)$

What’s the discontinuity?

- At any point $t$ where $F_X(u)$ is discontinuous $F_X(t^-) - F_X(t^+) = \text{jump in the value of } F_X(u)$
- $P(X = t) = \text{point mass at } t$
- At any point of $v$ where $F_X(u)$ is continuous $F_X(v) = F_X(v^+) = F_X(v^-)$
- $P(X = v) = 0$
Probabilities from the CDF

- By definition, \( F_X(u) = P(X \leq u) \)
- \( P(X = u) = F_X(u^+) - F_X(u^-) \)
  - height of jump (if any) at \( u \)
- \( P(X > u) = 1 - F_X(u) \)
- \( P(X \geq u) = 1 - F_X(u^-) \)
  - \( = 1 - F_X(u) \) if CDF is continuous
- \( P(X < u) = F_X(u^-) \)
  - \( = F_X(u) \) if CDF is continuous

More probabilities from the CDF

- By definition, \( F_X(u) = P(X \leq u) \)
- \( P(a < X \leq b) = F_X(b) - F_X(a) \)
- \( P(a \leq X < b) = F_X(b^-) - F_X(a^-) \)
- \( P(a < X < b) = F_X(b^-) - F_X(a) \)
- As in previous cases, the distinctions disappear at points of continuity of the CDF

Expectation of positive discrete RV

Consider a discrete random variable \( X \) taking on positive values \( a, b, c \) with probabilities \( p, q, r \) respectively; \( p + q + r = 1 \)

- \( E[X] = ap + bq + cr \)
Expectation of a discrete RV

\[ E[X] = -|a| \cdot p + c \cdot r + d \cdot t \]

- area above CDF to right of 0
- area below CDF to left of 0

area = |a| \cdot p
area = c \cdot r
area = d \cdot t

Expectation of an arbitrary RV

\[ E[X] = \int_{-\infty}^{\infty} [1 - F_X(u)] \, du - \int_{-\infty}^{\infty} F_X(u) \, du \]

| The expected value of an arbitrary random variable X can be defined as |

Summary

- We discussed the concept of the CDF of a random variable and its ramifications