Reminders

- Axiomatic theory of probability
- The set of all possible outcomes of an experiment is the sample space Ω
- Events are subsets of Ω
- An event is said to have occurred if the outcome of the experiment is a member of the event (that is, subset of Ω)
- A and A^c are a partition of Ω
- On every trial, one of A and A^c must occur

20-20 hindsight is a wonderful thing

- Once the experiment has been performed and the outcome is known, we have perfect and complete knowledge
- For each pair of events A and A^c, we can tell which one occurred and which one didn't — just check which set A or A^c the observed outcome belongs to!
- There is no probabilistic consideration any more, and we do not need to think about the chances of A (or A^c) occurring

A little learning is a dangerous thing..

- Now suppose that the experiment has been performed, but we do not know the outcome exactly
- All we know is that the outcome is some member of the event A, but we do not know which member of A it is
- Put another way, we are told that the event A has occurred, but nothing else
- To avoid trivial cases, assume that A is not a singleton or elementary event

… B or ¬B? That is the question…

- The experiment has been performed and we know that the event A has occurred, that is, the outcome is some member of A
- Did the event B occur? or did B^c occur?
- Unlike the case of perfect knowledge, we cannot tell whether B or B^c occurred
- AB and AB^c are a partition of A
- If the outcome AB, then B occurred: if the outcome AB^c, then B^c occurred

The exorcism did not work…

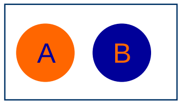
- The experiment has been performed and we know that the event A occurred, that is, the outcome is some member of A
- We cannot tell for sure whether B occurred
- We have not exorcised the probability from the problem as yet — A probability question still continues to plague us
- Question: What are the chances that B occurred? in view of the new knowledge that event A is known to have occurred?

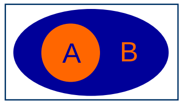
A mind like a steel trap…

- Question: What are the chances that B occurred? in view of the new knowledge that event A is known to have occurred?
- One (stupid?) answer to this question is that the chances that B occurred are still what they always were, viz. P(B)
- "Don't bother me with facts; my mind is made up!"
- This ostrich-like approach is wrong, at least in some cases

Refusing to face the facts…

- Question: What are the chances that B occurred? in view of the new knowledge that event A is known to have occurred?





- Left diagram: AB = . Obviously, if A occurred, B cannot have occurred
- Right diagram: AB = A. Obviously, if A occurred, B must also have occurred

Swaying every which way …

- More generally, even when the special cases AB = or AB = A do not hold, it is still reasonable to change the value of P(B) to reflect the information obtained from knowing that A occurred
- Probabilities are beliefs, and we can and should revise them as we grow wiser…
  Growing older is mandatory; growing wiser is optional
- We need some notation to distinguish between various values of P(B)

What's in a name?

- The (original) value of P(B) is called the unconditional or ***a priori*** probability of B
- Here ***a priori*** means "before the fact" or prior to the experiment being performed
- Given that A occurred, the revised chances of B occurring are called the

  conditional probability of B given A

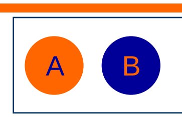
  and denoted by $P(B|A)$
- Read this as "probability of B given A"

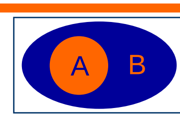
Definition of conditional probability

- The conditional probability of B given A is denoted by $P(B|A)$
- Read this as "the probability of B given A" or "the probability of B conditioned on A"
- A is called the conditioning event
- Definition: If $P(A) > 0$, $P(B|A)$ is defined as

$$P(B|A) = \frac{P(AB)}{P(A)}$$

Special cases





- Left diagram: AB = . Obviously, if A occurred, B cannot have occurred
- $P(B|A) = P(AB)/P(A) = 0$
- Right diagram: AB = A. Obviously, if A occurred, B must also have occurred
- $P(B|A) = P(AB)/P(A) = P(A)/P(A) = 1$

An example

- Example: Two fair dice are rolled. What is the probability that the sum of the two faces is 6 given that the dice are showing different faces?
- Critique of statement: Problems on conditional probability are often stated in the careless manner above. A conditional probability is being asked for, but the word conditional has been left out!

Help! Please explain more carefully

- Example: Two fair dice are rolled. What is the probability that the sum of the two faces is 6 given that the dice are showing different faces?
- How can we tell that we are to find a conditional probability in this problem?
- The phrasing … given that… **something happened** means that we are being told what the conditioning event is, and being asked for a conditional probability

Probability is so confusing…

- Example: Two fair dice are rolled. What is the probability that the sum of the two faces is 6 given that the dice are showing different faces?
- Distinguish the use of given that as asking for a conditional probability from given that as supplying basic data as in
  - Given that P(A) = 0.5, P(B) = 0.6 and P(AB) = 0.3, what is P(A B)?
  - No conditioning event is specified here

Enough already! Solve the example

- Example: Two fair dice are rolled. What is the probability that the sum of the two faces is 6 given that the dice are showing different faces?
- A= {dice are showing different faces}
  = Ω – {(1,1), (2,2), (3,3), (4,4), (5,5), (6,6)}
  P(A) = 30/36 = 5/6
- B= {sum is 6}
  = {(1,5), (2,4) ,(3,3), (4,2), (5,1)}
- AB = {(1,5), (2,4), (4,2), (5,1)}

Example (continued)

- Example: Two fair dice are rolled. What is the probability that the sum of the two faces is 6 given that the dice are showing different faces?
- P(A) = 30/36 = 5/6 • P(B) = 5/36
- AB = {(1,5), (2,4), (4,2), (5,1)}
- P(AB) = 4/36
- P(B|A) = P(AB)/P(A) = (4/36)/(30/36)
  = 2/15 < P(B)

Example (changed a bit)

- Example: Two fair dice are rolled. What is the probability that the sum of the two faces is 7 given that the dice are showing different faces?
- A= {dice are showing different faces}
  P(A) = 30/36 = 5/6
- C= {sum is 7}
  = {(1,6), (2,5) ,(3,4), (4,3), (5,2), (6,1)}
- P(C) = 6/36
- AC = C

Example (changed a bit, continued)

- Example: Two fair dice are rolled. What is the probability that the sum of the two faces is 7 given that the dice are showing different faces?
- P(A) = 30/36 = 5/6 • P(C) = 6/36
- AC = C
- P(C|A) = P(AC)/P(A) = (6/36)/(30/36)
  = 1/5 > P(C)

Example (changed a bit more)

- Example: Two fair dice are rolled. What is the probability that the first die is showing a 6 given that the dice are showing different faces?
- A= {dice are showing different faces}
  P(A) = 30/36 = 5/6
- D= {first die shows a 6}
  = {(6,1), (6,2) ,(6,3), (6,4), (6,5), (6,6)}
- P(D) = 6/36
- AD = {(6,1), (6,2) ,(6,3), (6,4), (6,5)}

Example (changed a bit more) ends

- Example: Two fair dice are rolled. What is the probability that the first die is showing a 6 given that the dice are showing different faces?
- P(A) = 30/36 = 5/6 • P(D) = 6/36
- AD = {(6,1), (6,2) ,(6,3), (6,4), (6,5)}
- $P(D|A) = P(AD)/P(A) = (5/36)/(30/36)$
  $= 1/6 = P(D)$

So, did we learn anything?

- The three different problems comprising the example showed that a conditional probability can be smaller, larger, or the same as, the unconditional probability
- They did?
- $P(B|A) = 2/15 < P(B) = 5/36$
- $P(C|A) = 1/5 > P(C) = 1/6$
- $P(D|A) = 1/6 = P(D) = 1/6$

Classical approach

- The definition of conditional probability is motivated by considerations arising from the classical probability viewpoint
- If $|\Omega| = n$ and $|A| = k$, then the reduced sample space consists of just the set A viewed as a sample space
- Outcomes in A have prob. 1/k, not 1/n
- What is the probability of B in this reduced sample space?

Reduced Sample Spaces

- Each outcome in A has probability 1/k (instead of 1/n)
- What is the probability of B in this reduced sample space?
- The only elements in B that are in A are members of AB whose size is $|AB|$
- "New" $P(B) = |AB|/k = (|AB|/n)/|A|/n$
  $= P(AB)/P(A) = P(B|A)$
- I hate reduced sample spaces!

Relative frequencies?

- The definition of conditional probability is motivated by considerations arising from the relative frequency viewpoint
- Suppose that N independent trials of the experiment have been performed
- Let N_A denote the number of trials on which event A occurred
- Let N_B denote the number of trials on which event B occurred

Focus on trials where A occurred…

- Events A and B respectively occurred on N_A and N_B trials out of N
- Event AB occurred on N_{AB} trials
- $N_A \approx P(A)\bullet N$ • $N_B \approx P(B)\bullet N$, etc
- Consider only those N_A trials on which A occurred and ignore the rest
- On how many of these did B also occur?
- If B occurred, then AB must have occurred (since A occurred on all these N_A trials)

You forgot a few, didn't you?

- Consider only those N_A trials on which A occurred and ignore the rest
- On how many of these did B also occur?
- If B occurred, then AB must have occurred (since A occurred on all these N_A trials)
- B occurred on N_{AB} trials out of the N_A trials on which A occurred
- The set of N_{AB} trials on which AB occurred must be a subset of the N_A trials on which A occurred

No, I didn't!

- Consider only those N_A trials on which A occurred and ignore the rest
- B occurred on N_{AB} trials out of the N_A trials on which A occurred
- The relative frequency of B on those N_A trials on which A occurred is N_{AB}/N_A
- $N_{AB}/N_A = (N_{AB}/N)/(N_A/N) \approx P(AB)/P(A)$ $= P(B|A)$
- Thus, conditional probability ≈ relative frequency on a restricted set of trials

Does it work for random variables?

- Can conditional probabilities be defined for random variables?
- Yes, we can find the conditional probability of an event defined in terms of random variables
- What is $P\{\mathbf{X} = k \mid \mathbf{X} > n\}$?
- This is just $P[\{\mathbf{X} = k\} \cap \{\mathbf{X} > n\}]/P\{\mathbf{X} > n\}$
- The event $\{\mathbf{X} = k\} \cap \{\mathbf{X} > n\}$ is either   or $\{\mathbf{X} = k\}$ depending on whether $k \le n$ or not

An example with random variables

- Let **X** denote a geometric random variable with parameter p
- For $k > 0$, $P\{\mathbf{X} = k+r \mid \mathbf{X} > r\}$

  $= P[\{\mathbf{X} = k+r\} \cap \{\mathbf{X} > r\}]/P\{\mathbf{X} > r\}$

  $= P\{\mathbf{X} = k+r\}/P\{\mathbf{X} > r\}$

  $= (1-p)^{k+r-1}\bullet p/(1-p)^r$

  $= (1-p)^{k-1}\bullet p$

  $= P\{\mathbf{X} = k\}$

Memoryless property I

- Let **X** denote a geometric random variable with parameter p
- For $k > 0$, $P\{\mathbf{X} = k+r \mid \mathbf{X} > r\} = P\{\mathbf{X} = k\}$
- Given that the event $\{\mathbf{X} > r\}$ has occurred, that is, the first r trials ended in a "failure", the probability that we need to wait for an additional k trials to observe the first success is the same as $P\{\mathbf{X} = k\}$
- It's as if the first r trials are forgotten!

Memoryless property II

- Geometric random variables are said to be memoryless in the sense that the waiting time for a success is unchanged by previous failures
- The chances of having the first success occur on the tenth trial from now are the same as they were 313 trials ago when we began the experiment
- The "system" has forgotten its past failures

Another example with RVs

- Let **X** denote a binomial random variable with parameters (n, p)
- A binomial random variable counts the number of occurrences of an event A of probability p on n independent trials
- Given that **X** = k, what is the conditional probability that the j-th trial resulted in a success? i.e., A occurred on the j-th trial?
- Let B = {**X** = k} and C = A on j-th trial

More on binomial RVs

- Let B = {**X** = k} and C = A on j-th trial
- $P(C|B) = P(BC)/P(B)$
- BC = {k total successes on n trials and a success on j-th trial}
  = {k–1 total successes on n–1 trials and a success on j-th trial}
- $P(BC) = \binom{n-1}{k-1} p^{k-1}(1-p)^{(n-1)-(k-1)} \cdot p$

More on binomial RVs (continued)

- Let B = {**X** = k} and C = A on j-th trial
- $P(C|B) = P(BC)/P(B)$
- $P(BC) = \binom{n-1}{k-1} p^{k-1}(1-p)^{(n-1)-(k-1)} \cdot p$
- $P(B) = \binom{n}{k} p^{k}(1-p)^{n-k}$
- $P(C|B) = P(BC)/P(B) = k/n$ = ratio of the binomial coefficients (proved earlier)

Partitions again

- AB and AB^c are a partition of A
- $P(AB) + P(AB^c) = P(A)$
- Dividing both sides by P(A), we get that

$$P(B|A) + P(B^c|A) = 1$$

that is,

$$P(B^c|A) = 1 - P(B|A)$$

- This is just like $P(B^c) = 1 - P(B)$ except that we are using conditional probabilities!

Just as good as regular probability

- Conditional probabilities are a probability measure, that is, they satisfy the axioms of probability theory
- Caveat: Everything must be conditioned on the same event. No mixing and matching allowed
- Thus, let A denote the fixed conditioning event and let all the probabilities under consideration be conditional probabilities

The axioms are satisfied …

- A is the conditioning event
- Axiom I: 0 ≤ P(B|A) ≤ 1 for all events B
  - Since AB   A, 0 ≤ P(AB) ≤ P(A) and hence 0 ≤ P(B|A) = P(AB)/P(A) ≤ 1
- Axiom II: P(Ω|A) = 1
  - Since AΩ = A, P(AΩ) = P(A) and hence P(Ω|A) = P(AΩ)/P(A) = 1
- Similarly for Axiom III

… and all the consequences hold

- P(|A) = 0
- P(B^c|A) = 1 – P(B|A)
- If B   C, then P(B|A) ≤ P(C|A)
- If BC =   , then

  P((B   C)|A) = P(B|A) + P(C|A)
- More generally,

  P((B   C)|A) = P(B|A) + P(C|A) – P(BC|A)

You say potato, I say potato…

- An expression such as P((B   C)|A) is commonly written as P(B   C|A)
- Everything to the right of the vertical bar is the conditioning event; it is a single set
- Everything to the left of the vertical bar is the conditioned event; it is a single set
- Beginners' mistake: If B and C are disjoint, they write P(B   C|A) = P(B) + P(C|A)
- NOT!

Keeping things straight

- Condition EVERYTHING on one event A and you can apply all the rules and tricks of probability that you have learned
- P(B   C|A   D) is the conditional probability of the event B   C conditioned on the event A   D
- Exercises: What is P(B|A   B)?

  What is P(A   B|A)? What is P(A^c B|A)?

  What is P(A^c B^c|A   B)?

Summary

- We studied the notion of conditional probability as a revised estimate of the chances that an event B occurred in light of partial knowledge of the outcome of the experiment, viz. knowing that A occurred
- Some simple examples were used to illustrate the concept
- We noted that conditional probabilities are a probability measure in that they satisfy the axioms of probability