Review of random variables

- A random variable **X** associates a number with each outcome of an experiment
- A random variable is a fixed map from the sample space to the real line
- Random because we do not know exactly which outcome of the experiment will be observed on the next trial, and thus which value the random variable will have
- Observed value of **X** varies at random

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Discrete random variables

- A discrete random variable takes on a finite number or a countably infinite number of discrete values
- The values taken on by a discrete random variable are discretely spaced
- If $u_1, u_2, \ldots$ are the values taken on by a discrete random variable, then for each choice of j, $u_j < u_{j+1}$

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Probability mass functions (pmfs)

- The probabilistic behavior of a discrete random variable **X** is described by its probability mass function $p_X(u)$ or $p(u)$
- $p_X(u) = p(u) = 0$ unless u is one of the values u_j that **X** takes on
- $p_X(u_j) = p(u_j) = P\{X = u_j\}$
- $p(u) \geq 0$ for all u; $\sum_j p(u_j) = 1$

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Everything you always wanted to know about X

- All the probabilistic information about the discrete random variable **X** is summarized in its pmf
- The pmf can be used to answer questions such as
  - "What is the probability that **X** has value between a and b?"
  - "What is the probability that **X** is an even number?"

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… but were afraid to ask!

- Example: **X** is a random variable taking on integer values 0 through 8
  $p_X(0) = p_X(1) = p_X(7) = p_X(8) = 0.05$
  $p_X(2) = p_X(3) = p_X(5) = p_X(6) = 0.15$; $p_X(4) = 0.2$
- $P\{3 < \mathbf{X} < 6\} = p_X(4) + p_X(5) = 0.35$
- $P\{3 \leq \mathbf{X} < 6\} = p_X(3) + p_X(4) + p_X(5) = 0.5$
- $P\{\mathbf{X} \text{ is odd}\} = p_X(1) + p_X(3) + p_X(5) + p_X(7) = 0.4$
- $P\{\mathbf{X} = 3.13\} = 0$ because $p_X(3.13) = 0$

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I'm leaving on a jet plane …

- Example (continued): Suppose that **X** is the number of passengers (with confirmed reservations) who show up for a flight on a 5-passenger plane. Let **Y** denote the number of passengers who board the flight Then, **Y** = **X** if **X** ≤ 5 and **Y** = 5 if **X** > 5
- The random variable **Y** is said to be a function of the random variable **X**
- The pmf of **Y** can be found from $p_X(u)$

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Why, oh why, oh why?

- Example (continued):
  $p_X(0) = p_X(1) = p_X(7) = p_X(8) = 0.05$
  $p_X(2) = p_X(3) = p_X(5) = p_X(6) = 0.15$; $p_X(4) = 0.2$
- **Y** = **X** if **X** ≤ 5; **Y** = 5 if **X** > 5
- $p_Y(0) = p_X(0) = 0.05$; $p_Y(1) = p_X(1) = 0.05$;
  $p_Y(2) = p_X(2) = 0.15$; $p_Y(3) = p_X(3) = 0.15$;
  $p_Y(4) = p_X(4) = 0.2$
  $p_Y(5) = p_X(5) + p_X(6) + p_X(7) + p_X(8) = 0.4$
- Sanity check: Is $\sum p_Y(u) = 1$? Yes

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Catching a few Z's

- Example (continued): For reasons of aircraft stability, control, and balance, the pilot insists that the plane must have an odd number of passengers. Let **Z** denote the number of passengers left behind (to sleep in the terminal while waiting for the next flight?)
- **Z** = 0 if **X** = 0, 1, 3, 5; **Z** = 1 if **X** = 2, 4, 6
  Z = 2 if **X** = 7; **Z** = 3 if **X** = 8

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The pmf of Z

- Example (continued):
  $p_X(0) = p_X(1) = p_X(7) = p_X(8) = 0.05$
  $p_X(2) = p_X(3) = p_X(5) = p_X(6) = 0.15$; $p_X(4) = 0.2$
- **Z** = 0 if **X** = 0, 1, 3, 5; **Z** = 1 if **X** = 2, 4, 6
  Z = 2 if **X** = 7; **Z** = 3 if **X** = 8
- $p_Z(0) = p_X(0) + p_X(1) + p_X(3) + p_X(5) = 0.4$;
  $p_Z(1) = p_X(2) + p_X(4) + p_X(6) = 0.5$;
  $p_Z(2) = p_X(7) = 0.05$; $p_Z(3) = p_X(8) = 0.05$
- Sanity check: Is $\sum p_Z(u) = 1$? Yes

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Repeated trials of the experiment

- Let **X** denote a discrete random variable with pmf $p_X(u)$
- **X** can take on values in $\{u_1, u_2, \ldots, u_n, \ldots\}$
- The experiment was repeated N times
- On these N trials, **X** was observed to have taken on values $x_1, x_2, \ldots, x_N$ respectively
- Each x_i is some number in the set $\{u_1, u_2, \ldots, u_n, \ldots\}$. It is possible that the same value is observed more than once

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Average of the N observed values

- On N trials, **X** was observed to have taken on values $x_1, x_2, \ldots, x_N$ respectively
- The average of these N numbers is just
  $$\frac{x_1 + x_2 + \ldots + x_N}{N}$$
- If we were to do another set of N trials, the values taken on by **X** on these trials would not be the same as $x_1, x_2, \ldots, x_N$
- But, the average would be fairly close!

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Repeated additions = multiplication!

- On N trials, **X** was observed to have taken on values $x_1, x_2, \ldots, x_N$ respectively
- The average of these N numbers is just
  $$\frac{x_1 + x_2 + \ldots + x_N}{N}$$
- Some of the x_i happen to be u_1, others are u_2, still others are u_3, and so on
- If a_1 of the x_i happen to have value u_1, they will contribute a total of a_1u_1 to the sum in the numerator

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The sum of identical numbers

- On N trials, **X** was observed to have taken on values $x_1, x_2, \ldots, x_N$ respectively
- The average of these N numbers is just
$$\frac{x_1 + x_2 + \ldots + x_N}{N}$$
- The average can also be expressed as
$$\frac{a_1u_1 + a_2u_2 + \ldots + a_nu_n + \ldots}{N}$$
where for each k, u_k occurred a_k times among the N numbers being added

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Where's the probability?

- On N trials, **X** was observed to have taken on values $x_1, x_2, \ldots, x_N$ respectively
- What does probability mean? On N trials, an event of probability p is expected to occur roughly pN times
- The event $\{\mathbf{X} = u_k\}$ is expected to occur roughly $P\{\mathbf{X} = u_k\}N = p_{\mathbf{X}}(u_k)N$ times
- Moral: We expect that a_k, the number of times that X had value u_k on the N trials, is approximately $p_{\mathbf{X}}(u_k)N$

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…and the average is approximately

- On N trials, **X** was observed to have taken on values $x_1, x_2, \ldots, x_N$ respectively
- The average of these numbers is
$$\frac{a_1u_1 + a_2u_2 + \ldots + a_nu_n + \ldots}{N}$$
where for each k, u_k occurs a_k times among the N numbers being added
- $\sum a_k = N$. Also, for each k, $a_k \approx p_{\mathbf{X}}(u_k)N$
- Moral: The average value of **X** over many trials is approximately $\sum u_k \bullet p_{\mathbf{X}}(u_k)$

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Definition of expectation

- The average value of the discrete random variable **X**, averaged over many trials, is approximately $\sum u_k \bullet p_{\mathbf{X}}(u_k)$
- We cannot guarantee that the average value will be exactly this number, but we expect it to be fairly close
- This notion is enshrined in probability theory by defining the expectation of **X** as $E[\mathbf{X}] = \sum u_k \bullet p_{\mathbf{X}}(u_k)$

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I say what I mean …

- The expectation of the discrete random variable **X** is given by
$$E[\mathbf{X}] = \sum u_k \bullet p_{\mathbf{X}}(u_k)$$
where the sum is over all values of **X**
- Read as "E of **X**" or "E **X**"
- Other names for expectation are
  - the expected value of **X**
  - the mean or mean value of **X**
  - the average or the average value of **X**

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… and I mean what I say

- The expectation of the discrete random variable **X** is
$$E[\mathbf{X}] = \sum u_k \bullet p_{\mathbf{X}}(u_k)$$
- The operational meaning of E[**X**] is exactly what we have discussed in this lecture: E[**X**] is approximately what we expect to observe as the average value of **X** over many trials
- But there are no guarantees

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Easy examples I

- Consider an experiment of rolling a fair die
- If the outcome is {1, 4}, you win \$1; if the outcome is {2, 3, 5, 6}, you lose \$1
- The random variable **X** that denotes your winnings takes on values +1 and –1
- $p_X(u) = 1/3$ if $u = +1$; $p_X(u) = 2/3$ if $u = -1$
- $E[X] = (+1)\bullet(1/3) + (-1)\bullet(2/3) = -1/3$
- Note that in this experiment, **X** will always differ from its mean E[**X**] on every trial

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Easy examples II

- Example: **X** is a Bernoulli random variable with parameter p, $0 < p < 1$
- $p_X(u) = p$ if $u = 1$; $p_X(u) = 1-p$ if $u = 0$
- $E[X] = 0\bullet(1-p) + 1\bullet(p) = p$
- Here too, **X** can never be equal to its mean on any trial

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Easy examples III

- Example: **X** is a random variable taking on integer values 0 through 8
  $p_X(0) = p_X(1) = p_X(7) = p_X(8) = 0.05$
  $p_X(2) = p_X(3) = p_X(5) = p_X(6) = 0.15$; $p_X(4) = 0.2$
- $E[X] = 0\bullet(0.05) + 1\bullet(0.05) + \ldots + 8\bullet(0.05)$
  $= 4$
- In this case, **X** can equal its mean on some trials (in fact, **X** = E[**X**] on roughly 20% of the trials)

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The moment about the origin

- The expectation of the discrete random variable **X** is

$$E[X] = \sum u_k \bullet p_X(u_k)$$

- The pmf defines a set of point masses
- The point mass $p_X(u_k)$ is at distance u_k from the origin and has a moment of $u_k \bullet p_X(u_k)$ about the origin
- Total moment of all the point masses is E[**X**], the expectation of **X**

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Location of the center of mass

- Total moment of all the point masses is just E[**X**], the expectation of **X**
- Center of mass is defined by the equation

Total moment = total mass × center of mass

- But total probability mass is 1
- Hence, E[**X**] = location of center of mass
- There need not be any actual mass at the center of mass
- Practical example: consider a doughnut!

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Fair entry fee for playing the game

- You stake \$1 on the roll of a fair die
- If the outcome is {1, 4}, you get \$2 back if the outcome is {2,3,5,6}, you get \$0 back
- The random variable **Z** denotes the return on your stake or investment
- $E[Z] = (2)\bullet(1/3) + (0)\bullet(2/3) = 2/3$
- \$1 stake means a loss of \$1/3 on average
- A fair entry fee for playing this game (that is, your stake) would be \$2/3

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How many passengers get to fly?

- Example (continued): **Y** denotes the number of passengers who board the flight. Then, **Y** = **X** if **X** ≤ 5 and **Y** = 5 if **X** > 5
- $p_Y(0) = p_Y(1) = 0.05$; $p_Y(2) = p_Y(3) = 0.15$; $p_Y(4) = 0.2$; $p_Y(5) = 0.4$
- $E[Y] = 0\bullet p_Y(0) + 1\bullet p_Y(1) + 2\bullet p_Y(2) + 3\bullet p_Y(3) + 4\bullet p_Y(4) + 5\bullet p_Y(5) = 3.6$
- On average, 4 passengers show up for the flight but only 3.6 get to fly!

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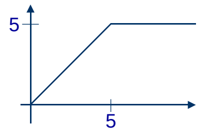
Y is a function of X

- Example (continued) **Y** is a function of **X** given by **Y** = **X** if **X** ≤ 5 and **Y** = 5 if **X** > 5
- We can write **Y** = g(**X**) where g(•) is the function shown below



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E[Y] revisited

- $p_Y(0) = p_X(0)$; $p_Y(1) = p_X(1)$; $p_Y(2) = p_X(2)$; $p_Y(3) = p_X(3)$; $p_Y(4) = p_X(4)$;
  $p_Y(5) = p_X(5) + p_X(6) + p_X(7) + p_X(8)$
- $E[Y] = 0\bullet p_Y(0) + 1\bullet p_Y(1) + 2\bullet p_Y(2) + 3\bullet p_Y(3) + 4\bullet p_Y(4) + 5\bullet p_Y(5)$
  $= 0\bullet p_X(0) + 1\bullet p_X(1) + 2\bullet p_X(2) + 3\bullet p_X(3) + 4\bullet p_X(4) + 5\bullet[p_X(5) + p_X(6) + p_X(7) + p_X(8)]$
  $= g(0)\bullet p_X(0) + g(1)\bullet p_X(1) + g(2)\bullet p_X(2) + g(3)\bullet p_X(3) + g(4)\bullet p_X(4) + g(5)\bullet p_X(5) + g(6)\bullet p_X(6) + g(7)\bullet p_X(7) + g(8)\bullet p_X(8)$

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Conclusion from the messy formula

- To find E[**Y**], where **Y** = g(**X**), we can find the pmf of **Y** from the pmf of **X**, and then use the standard result that
  $$E[Y] = \sum v_j \bullet p_Y(v_j)$$
  where sum is over all values v_j of **Y**
- Alternatively, we can find E[**Y**] via the formula $E[Y] = E[g(X)] = \sum g(u_k)\bullet p_X(u_k)$ where the sum is over all values u_k of **X**
- The new formula is just a re-arrangement of the terms in the standard sum for E[**Y**]

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It bears repeating that…

- The standard definition $E[Y] = \sum v_j \bullet p_Y(v_j)$ requires finding/knowing the pmf of **Y**
- To find E[**Y**], where **Y** = g(**X**), it is not necessary to first find the pmf of **Y** from the pmf of **X**. Instead, we can use
  $$E[Y] = E[g(X)] = \sum g(u_k)\bullet p_X(u_k)$$
  where the sum is over all values u_k of **X**
- We have avoided the extra step of finding $p_Y(v_j)$ to use in the standard definition

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Expected value of a function of X

- If Y = g(**X**), the expected value of **Y**, which is given by $E[Y] = \sum v_j \bullet p_Y(v_j)$ can often be more easily computed as
  $$E[Y] = E[g(X)] = \sum g(u_k)\bullet p_X(u_k)$$
  where the sum is over all values u_k of **X**
- It is not necessary to find $p_Y(v_j)$ to use in the standard definition of E[**Y**]; the pmf of **X** and the function g(•) suffice
- This theorem is called LOTUS
- LOTUS makes it easy to find E[**Z**] (slide 8)

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Political correctness and LOTUS

- Many statistics texts define $E[g(\mathbf{X})]$ as
  $$E[g(\mathbf{X})] = \sum g(u_k) \cdot p_{\mathbf{X}}(u_k)$$
  where the sum is over all values u_k of $\mathbf{X}$, apparently without realizing that $g(\mathbf{X})$ is a random variable $\mathbf{Y}$, and hence its expected value is, by definition, $E[\mathbf{Y}] = \sum v_j \cdot p_{\mathbf{Y}}(v_j)$
- That both computations give the same result is a theorem of probability theory
- The first four editions of Ross call this the Law of the Unconscious Statistician

Applications of LOTUS I

- $\mathbf{Y} = a\mathbf{X} + b$, where a and b are constants.
- $E[\mathbf{Y}] = E[g(\mathbf{X})] = \sum g(u_k) \cdot p_{\mathbf{X}}(u_k)$
  $= \sum (a \cdot u_k + b) \cdot p_{\mathbf{X}}(u_k)$
  $= a \cdot \sum u_k \cdot p_{\mathbf{X}}(u_k) + b \cdot \sum p_{\mathbf{X}}(u_k)$
  $= a \cdot E[\mathbf{X}] + b$
- $E[a\mathbf{X} + b] = a \cdot E[\mathbf{X}] + b$
- Expectation is a linear operation: the expectation of a sum is the sum of the expectations
- $E[b] = b$

Applications of LOTUS II

- Suppose that $\mathbf{Y} = \mathbf{X} - a$, where a is some constant. Then,
- $E[\mathbf{Y}] = E[\mathbf{X}] - a$
- The mass $p_{\mathbf{X}}(u_k)$ is at distance $u_k - a$ from the point a on the real line
- $(u_k - a) \cdot p_{\mathbf{X}}(u_k)$ is the moment of the mass $p_{\mathbf{X}}(u_k)$ about the point a on the real line
- $E[\mathbf{X} - a]$ = total moment about a

Applications of LOTUS III

- $E[\mathbf{X} - a]$ = total moment about the point a
- If we choose a to be $E[\mathbf{X}]$, then we get that $E[\mathbf{X} - a] = E[\mathbf{X}] - a = E[\mathbf{X}] - E[\mathbf{X}] = 0$
- $E[\mathbf{X}]$ is the center of mass
- The moment about the center of mass is 0
- A body is "perfectly balanced" about its center of mass
- The quantity $\mathbf{X} - E[\mathbf{X}]$ is the deviation of $\mathbf{X}$ from its mean
- The average deviation from the mean is 0

Applications of LOTUS IV

- The average deviation from the mean is 0
- We shall often use μ or $\mu_{\mathbf{X}}$ to denote the mean, that is, expected value of $\mathbf{X}$
- In probability theory as well as in many other areas, the average value of the squared deviation from μ, that is, the expected value of $(\mathbf{X} - \mu)^2$ is important
- $E[(\mathbf{X} - \mu)^2]$ is called the variance of $\mathbf{X}$
- Variance is denoted by $\mathrm{var}(\mathbf{X})$ or σ^2 or $\sigma_{\mathbf{X}}^2$ where σ is called the standard deviation

The variance of a random variable

- $\sigma^2 = E[(\mathbf{X} - \mu)^2]$ is called the variance of $\mathbf{X}$
- LOTUS tells us that $\sigma^2 = E[(\mathbf{X} - \mu)^2] = \sum (u_k - \mu)^2 \cdot p_{\mathbf{X}}(u_k)$
- $\sigma^2 \geq 0$ since all the terms in the sum are nonnegative
- In fact, $\sigma^2 > 0$ except when the random variable happens to be a constant!
- If your computations give you negative variance or zero variance, you are likely to have made a mistake! Check your work!

More on the variance

- $\sigma^2 = E[(\mathbf{X} - \mu)^2]$ is called the variance of **X**
- LOTUS tells us that
  $\sigma^2 = E[(\mathbf{X} - \mu)^2] = E[\mathbf{X}^2 - 2\mu\mathbf{X} + \mu^2]$
  $= E[\mathbf{X}^2] - E[2\mu\mathbf{X}] + E[\mu^2]$
  $= E[\mathbf{X}^2] - 2\mu \cdot E[\mathbf{X}] + \mu^2$
  $\sigma^2 = E[\mathbf{X}^2] - \mu^2$
- Alternatively, expand out $\sum(u_k - \mu)^2 \cdot p_{\mathbf{X}}(u_k)$ and verify that you get the same result
- $\sigma^2 = E[(\mathbf{X} - \mu)^2] = E[\mathbf{X}^2] - \mu^2$
  $= E[\mathbf{X}^2] - (E[\mathbf{X}])^2$ or $E[\mathbf{X}^2] - E^2[\mathbf{X}]$

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What does the variance represent?

- $\sigma^2 = E[(\mathbf{X} - \mu)^2]$ is called the variance of **X**
- $\sigma^2 = \sum(u_k - \mu)^2 \cdot p_{\mathbf{X}}(u_k)$
- The point mass $p_{\mathbf{X}}(u_k)$ is at distance $u_k - \mu$ from the center of mass
- Hence, this point mass has moment of inertia $(u_k - \mu)^2 \cdot p_{\mathbf{X}}(u_k)$ about the center of mass
- σ^2 denotes the moment of inertia about the center of mass
- σ is also the radius of gyration

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What does large variance mean?

- σ^2 denotes the moment of inertia about the center of mass
- σ is also the radius of gyration: the system of point masses can be represented by a two half-unit masses at locations $\mu \pm \sigma$
- Since the total mass is always 1, a large variance means the total mass is spread widely and far away from the mean
- Small variance means mass is close to μ
- A zero variance means all the mass is at μ

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Summary

- How to calculate probabilities of various interesting events from the pmf
- How to find the pmf of a function g(**X**)
- The average value of **X** on repeated trials
- E[X], the expectation of **X**
- Interpretations of E[X]
- Finding E[g(**X**)] via LOTUS
- Variance of a random variable
- Properties of the variance

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