Topics studied in previous class

- Sampling without replacement and its relation to random samples
- Sample spaces with countably infinite outcomes
- Axiom III for countably infinite disjoint sets
- Relative frequencies; many trials
- Uncountably infinite sample spaces
- Real numbers versus reality

Measurements are rational

- A physical measurement made with an instrument yields a rational number
- At the microscopic level, most physical phenomena are discrete
- Any electrical charge is an integer multiple of the electrical charge of an electron
- Electrical current (charge/unit time) is a rational multiple of the quantity one electron charge/s

But, the models are real numbers

- The `real` numbers in computer programs are actually rational numbers — irrational numbers cannot be represented exactly
- Then, why are physical parameters modeled as continuous variables?
- Easier to get (the “right”) answers
- Calculus can be used: $L\frac{di}{dt} + Ri = v$

  assumes that i is a continuous function of t

Measurements versus models

- All actual measurements result in rational numbers, but we model the measurement as being an arbitrary real number
- We all understand that V = 1.235 volts really means that V is some real number in the range $1.2345 \le V \le 1.2355$ volts
- This modeling is useful and convenient in the physical sciences and engineering

  but causes difficulties in probability theory

Real numbers in probability

- Uncountably infinite sample spaces are the real line (or intervals thereof)
- Such spaces cause subtle mathematical difficulties in probability theory
- The resolution of these difficulties led to the development of the axiomatic theory of probability
- We will look only at the end results, not the gory details

Relative frequencies converge to 0

- The output of **`rand()`** is a good model for repeated trials for the experiment of picking a number at random in (0, 1)
- Every call to **`rand()`** returns a different number
- Relative frequency of a specific number in (0,1) converges to 0
- Actually, the output of **`rand()`** is periodic (with long period,) and the numbers will repeat

P{outcome} is always 0

- The only model that works for uncountably infinite sample spaces is for each outcome to have probability 0
- But, on each trial, some outcome occurs
- Moral: An event whose probability is zero can occur
- Complementary event has probability one
- Immoral: An event whose probability is one need not occur on every trial!

Events of probability zero

- A event of probability zero is one to which we assign a probability of 0
- A event of probability zero is not the same as , the impossible event
- A zero-probability event can occur on a trial of the experiment: never ever does
- It is just that a zero-probability event does not occur too often — in fact, it is usually observed only once at most

Events of probability one

- The complement of an event of probability zero is called an event of probability one
- An event of probability one is not the same as Ω, the certain or sure event
- An event of probability one need not occur on every trial: Ω occurs on all trials
- An event of probability one is called an almost sure event because it almost always occurs

Any probabilities other than 0 or 1?

- If every outcome is an event of probability zero, then isn't it true that any event A must also have probability zero?
- P(A) = sum of the probabilities of all the outcomes that comprise A
  = 0 + 0 + … = 0?
- No, the above is a mis-application of Axiom III (which applies to countable unions only)

P(countable event) = 0

- Since each outcome has probability zero, a countable event, that is, an event that has a countable number of outcomes, also has probability zero (by Axiom III)
- Axiom III does not say that the probability of an uncountable event is the sum of the probabilities of the outcomes
- The nonzero (and non-one) probabilities are assigned to such uncountable events

A probability paradox

- Example: Choose a random number between 0 and 1
- The rational numbers between 0 and 1 are a countable set
- P(outcome is a rational number) = 0
- But, in any simulation of this experiment, e.g. via calls to `rand()`, the "outcome" will be a rational number
- Remember that `rand()` is a model

Intervals are uncountable events

- Example: Choose a random number between 0 and 1
- Each outcome (and also any countable set of outcomes) has probability zero
- However, P{a < outcome < b} = b – a for 0 ≤ a < b ≤ 1
- The nonzero probabilities are assigned to the intervals of the line, not to outcomes!

Asking the right question

- In most physical applications, the question “Does x = 0.213482774099070267623…?” is meaningless
- If x were 0.213482774099070267624… instead, the airplane will still fly, the bridge will still stand, the modem will still connect
- In most instances, we are satisfied if x is in some specified range (design specs)
- “Does x (a,b)?” is the right question!

Intervals have nonzero probabilities

- Example: Choose a random number between 0 and 1
- P{a < outcome < b} = b–a for 0 ≤ a < b ≤ 1
- P{0.4 < outcome < 0.6} = 0.2
- N calls to `rand()` give N numbers, roughly 20% of which are in the interval (0.4, 0.6)
- At most one (and most likely none!) of these will be 0.57689231

What about P(arbitrary subset)?

- For uncountably infinite sample spaces, a consistent probability assignment to all the subsets of Ω is not possible
- If we restrict the class of subsets of Ω to which we will assign probabilities, then a consistent assignment of probabilities is possible
- It is meaningless to talk of probabilities of subsets that are not in this restricted class

Rules for collections of events

- Let denote the collection of subsets of Ω that we will call the events and to which we will assign probabilities
- The members of are subsets of Ω
- Rule I: Ω
- Rule II: If A , then A^c also
- Rule III: If A_1, A_2, … A_n, … is a countable sequence of events, then, A_1 A_2 … A_n … also

The σ- field of events

- A collection of subsets of Ω that satisfies Rules I–III is called a σ-field
- Rules I–III can be summarized as follows: A σ-field contains Ω and is closed under complementation and countable unions
- “Closed under” means the result of the specified operation also belongs to
- By DeMorgan’s theorem, the σ-field is also closed under countable intersections

Small σ- fields

- For a finite or countably infinite sample space, the collection of all the subsets of Ω is a σ-field
- If ï Ωï = n, this σ-field has 2^n events in it
- Smaller σ-fields also exist:
  {Ω, } is a σ-field as is {Ω, A, A^c, }
- Given any partition of Ω, the collection of all the sets that can be written as the union of the sets in the partition is also a σ-field

The σ- field of the real line

- When Ω is the real line (or an interval thereof), we have a Rule IV for the σ-field
- Rule IV: The σ-field contains all intervals of the form (a, b) with a < b
- It can be shown that Rules I–III imply that intervals of the form
  - [a, b), (a, b], and [a, b]
  - (∞, a), (∞, a], (a, ∞), and [a, ∞)

  also belong to the σ-field

More on the σ- field of the real line

- The σ-field of the real line contains all intervals of all types (open, closed etc)
- Since we will assign probabilities only to the members of the σ-field, this ensures that all the right engineering questions such as
  "What is P{0.23 < outcome < 0.25}?"
  "What is P{0.23 ≤ outcome ≤ 0.29}?"
  have useful answers

Not in the σ- field of the real line?

- The σ-field of the real line contains all the intervals of all types
- It contains the complements and the countable unions and intersections of these intervals
- Are there subsets of the real line that are not of this type? Yes
- Professor, can you describe one to us?
- Need Math 441 to understand description

The Probability Space

- The probability space is the formal statement of the axiomatic theory
- A probability space (Ω, , P) consists of
  - the sample space Ω consisting of all possible outcomes of the experiment
  - the σ-field of events which includes all of the interesting subsets of Ω
  - the probability measure P(•) that assigns probabilities to the events in

The Probability Space (continued)

- The probability measure P(•) assigns probabilities to the events in subject to the following rules (axioms)
- Axiom I: P(A) ≥ 0 for all events A
- Axiom II: P(Ω) = 1
- Axiom III: If $A_1, A_2, \ldots A_n, \ldots$ is a countable sequence of disjoint events, then $P(A_1 \quad A_2 \quad \ldots \quad A_n \quad \ldots)$
  $= P(A_1) + P(A_2) + \ldots + P(A_n) + \ \ldots$

Footnote to Probability Space

- If Ω is the real line, then we assume that , the σ-field of events, consists of all the open intervals, and all the other events that it must contain as per Rules I–III
- In this case, also contains all semi-closed and closed intervals as well
- There do exist weird subsets of Ω that are not in and do not have probabilities
- These subsets never arise in applications

Duh! So what is all this good for?

- The axiomatic approach is the foundation
- The consequences of the axioms have already been looked at earlier
- In practice, the formal axiomatic approach to probability is not used on a everyday basis in solving problems
- It is important to know what are the right questions to ask: In infinite sample spaces, ask for P{a < x < b} and not for P{x = c}

We know everything already

- The axiomatic theory tells about the probability measure P(•)
- Since we know P(•), what is left to study?
- Generally, P(•) is known for only some of the events
- The probability calculus allows us to calculate the probabilities of other events
- So, why don't we estimate these other probabilities via (say) relative frequencies?

We don't know everything already

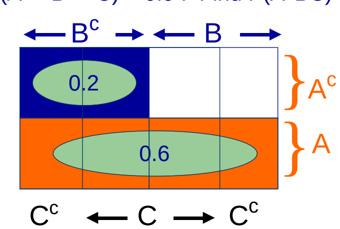
- So, why don't we estimate these other probabilities via (say) relative frequencies?
- Some probabilities may be too expensive or too small to estimate
  - reliability of complex systems is more easily calculated than measured
  - how do we measure the probability that a nuclear reactor has a meltdown?
- Calculate probabilities of complex events from probabilities of simpler events

An example

- Example: Events A, B, C are defined on a sample space Ω. Given that P(A) = 0.6, P(A B^c) = 0.8 P(A B C) = 0.9, and P(A^cC) = 0.15, find P(A^cBC)
- As before, the first step is to draw a Karnaugh map

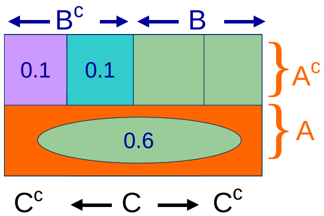
Find P(A^cBC)

- P(A) = 0.6, P(A B^c) = 0.8, P(A^cC) = 0.15 P(A B C) = 0.9 . Find P(A^cBC)



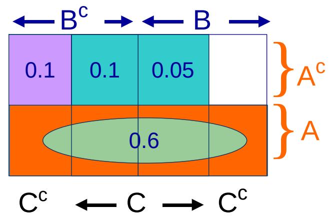
Find $P(A^cBC)$

- $P(A) = 0.6$, $P(A \quad B^c) = 0.8$, $P(A^cC) = 0.15$
  $P(A \quad B \quad C) = 0.9$. Find $P(A^cBC)$



Find $P(A^cBC)$

- $P(A) = 0.6$, $P(A \quad B^c) = 0.8$, $P(A^cC) = 0.15$
  $P(A \quad B \quad C) = 0.9$. Find $P(A^cBC)$



Summary

- We discussed measurements vs models
- We studied uncountably infinite sample spaces and noted P{outcome} = 0
- We discussed the restricted notion of events in uncountably infinite sample spaces
- We gave a formal statement of the concept of a probability space
- We did an example

