Review of Lecture #1

- Intuitive ideas of probability and statistics were introduced
- Various approaches to probability were described briefly
- The notion of statistical inference was introduced
- But what does all this have to do with electrical and computer engineering?

Probability in Physics

- Maxwell-Boltzmann kinetic theory of gases
  - Motion of gas molecules is random
  - Gas laws can be deduced from the aggregate behavior of many molecules
- Statistical mechanics
- Quantum mechanics
  - At the atomic level, physical phenomena can only be described probabilistically

Probability in Physics

- Quantum mechanical view of probability is somewhat different
- Semiconductor physics and electronics
  - Electron clouds: Fermi-Dirac statistics instead of Maxwell-Boltzmann statistics
  - Many practical devices in use these days are designed to make use of quantum-mechanical effects

Probability in Engineering

- Thermal noise in electrical circuits
- Detection of weak radio and radar signals
- Information theory
- Communication systems design
- Reliability of systems
  - Failure probabilities
  - Failure rates
  - Mean time to failure

Probability in Engineering

- Networks and Systems Problems
  - Random arrivals of packets/jobs
  - Random lengths/service times
  - Random requests for resources
  - Probability of buffer or queue overflow
  - Transmission or service delays
  - Scheduling problems, priorities, QOS
  - Flow control and routing

Probability in Engineering

- So many problems, so little time …
- In this course, we emphasize the basic science and math
- Applications are generally found in the examples, and use over-simplified models
- You need to take follow-on courses to gain better understanding of where and how probability is used in engineering

Experiments and Trials

- Fundamental notion: An experiment is performed and its outcome observed
- This is called a trial of the experiment
- The experiment may be performed by a human agent, e.g. tossing a coin
  or rolling a die
- The epxperimental outcome might just be the measurement of a naturally occuring random phenomenon, e.g. a noise voltage

The Sample Space

- The set of all possible outcomes of an experiment is called the sample space Ω of the experiment
- Example: The experiment is tossing a coin: $\Omega = \{ H, T \}$
- Example: The experiment is rolling a die: $\Omega = \{ 1, 2, 3, 4, 5, 6 \}$
- Example: The experiment is measuring a noise voltage: $\Omega = \{ x : -1 \le x \le 1 \}$

A simple probability assignment

- Example: The experiment is rolling a die: $\Omega = \{ 1, 2, 3, 4, 5, 6 \}$
- Suppose that each outcome is equally likely: $P(1) = P(2) = \ldots = P(6) = 1/6$
- What is the probability of rolling an even number?
- 1/2
- What is the probability of rolling a prime number? 1/2 also (1 is not a prime)

More on probabilities

- An even number is said to have been rolled if the outcome is any of {2, 4, 6}
- P(even number) = 1/2; more explicitly P(even number) = 3/6 since 3 of the 6 outcomes are in the subset {2, 4, 6}
- A prime number is said to have been rolled if the outcome is any of {2, 3, 5}
- P(prime number) = 3/6 = 1/2 also

What is an Event?

- A subset of Ω is called an event
- Example: $A = \{2, 4, 6\}$ and $B = \{2, 3, 5\}$ are said to be events defined on the sample space $\Omega = \{1, 2, 3, 4, 5, 6\}$
- “events defined on the sample space” is merely a probabilist’s way of saying “subsets of the sample space”
- $A^c = \{1, 3, 5\}$ and $B^c = \{1, 4, 6\}$ also are events defined on Ω

When does an event occur?

- An event A is said to have occurred on a trial if the outcome of the trial is a member of the subset A
- Event A occurs if the observed outcome is some member of A; we don’t care which member of A it is
- If the observed outcome is not a member of A, then we say A did not occur, or equivalently, we say that A^c occurred

Outcomes versus Events

- Every trial results in only one outcome, that is, only one of the elements in Ω can be the observed outcome
- The observed outcome is a member of several different subsets, i.e., events, and all these events are said to have occurred
- Fundamental notion: On each trial of the experiment, one outcome occurs, but many events occur

Many events occur

- Example: If the outcome of rolling a die is 4, then
  - Events $A = \{2, 4, 6\}$ and $B^c = \{1, 4, 6\}$ both have occurred
  - Events $A^c = \{1, 3, 5\}$ and $B = \{2, 3, 5\}$ did not occur
  - Event A $B^c = \{1, 2, 4, 6\}$ has occurred
  - Event $A \cap B^c = \{4, 6\}$ also has occurred

Elementary or singleton events

- An event containing a single outcome is called an elementary event or singleton event
  - $A \cap B = \{2\}$ is an elementary event
- We abuse notation and use singleton events and outcomes interchangeably
- Nitpicking: 4 is a member of Ω: $4 \; \Omega$ while {4} is a subset of Ω: $\{4\} \; \Omega$

Outcomes versus Events Revisited

- Every trial results in only one outcome, say x is the outcome
- Fundamental notion: On each trial of the experiment, one outcome (x in this case) occurs, but many events occur
- Of course, only the one elementary event {x} has occurred on this trial; all the other many events that have occurred have {x} as a subset, that is, A occurred iff $\{x\} \quad A$

Events occur/do not occur in pairs

- Suppose that a trial resulted in outcome x
- Exactly one elementary event, viz. {x} occurred on this trial
- Many (non-elementary) events occurred on this trial
- For any event A, exactly one of the two events A and A^c occurred, and the other did not

One of A and A^c always occurs

- Example: If the outcome of rolling a die is 4, then events
  - Event $A = \{2, 4, 6\}$ occurred; event $A^c = \{1, 3, 5\}$ did not occur
  - Event $B^c = \{1, 4, 6\}$ occurred; event $B = \{2, 3, 5\}$ did not occur
- If the outcome has been 2 instead, then A and B would have occurred, and A^c and B^c would not have occurred

Two special events

- Ω can be regarded as a subset of Ω
- On any trial, the event Ω always occurs
- The event Ω is called the certain event or the sure event
- , the empty set, is also a subset of Ω
- On any trial, the event never occurs
- The event is called the null event or the impossible event

How many events are there in all?

- A sample space Ω of n elements has 2^n different subsets including Ω and
- $2^n – 1$ of these subsets are nonempty
- The 2^n events can be paired up into $2^{n–1}$ pairs of the form {A, A^c}
- Special case: {Ω, } is one such pair
- On each trial of the experiment, exactly $2^{n–1}$ events (one from each pair) occur and $2^{n–1}$ events (other from each pair) do not

Probabilities of the special events

- Ω always occurs; $Ω^c$ = never occurs
- Obvious conclusion: the probabilities assigned to Ω and should be 1 and 0 respectively, regardless of how we choose to assign probabilities to the outcomes
- P(Ω) = 1 will be used as an axiom in the axiomatic approach to probability
- What are the probabilities of the other events?

Classical probability assignment

- For sample spaces with n elements $x_1, x_2, …, x_n$, the probabilities of events depend on the probabilities of the outcomes
- Classical approach: Each outcome has probability 1/n; $P(x_i)$ or $P(\{x_i\})$ = 1/n for all i
- P(A) = ½A½/n ½A½= # of elements in A
- Special cases: P(Ω) = n/n = 1
  P( ) = 0/n = 0

Arbitrary probability assignment

- Nonclassical approach: The n outcomes have probabilities $p_1, p_2, …, p_n$ where $p_i \geq 0$ and $\sum p_i = 1$
- The probability of an event A is the sum of the probabilities of all the outcomes that comprise A
- P(A) = sum of the p_i for all members of A
- Example: A = $\{x_2, x_4, x_{22}\}$
  P(A) = $p_2 + p_4 + p_{22}$

Extension of classical approach

- Example: A = $\{x_2, x_4, x_{22}\}$
  P(A) = $p_2 + p_4 + p_{22}$
- Special cases: P( ) = 0 as before
  P(Ω) = $p_1+p_2+ …+p_n$ = 1
- The nonclassical approach reduces to the classical approach if all the p_i = 1/n.

  P(A) = (1/n+1/n+…1/n) for ½A½outcomes, i.e. P(A) = ½A½/n

Disjoint Events

- Events A and B are said to be disjoint or mutually exclusive if A ∩ B =
- A and B have no element in common
- A = $\{x_2, x_4, x_{22}\}$ and B = $\{x_3, x_7, x_9\}$ are disjoint events
- A B = $\{x_2, x_3, x_4, x_7, x_9, x_{22}\}$
- P(A B) = $p_2 + p_3 + p_4 + p_7 + p_9 + p_{22}$ = P(A) + P(B)

Probability of a Disjoint Union

- If events A and B are disjoint, then A B is said to be a disjoint union of events
- For a disjoint union of events A and B, P(A B) = P(A) + P(B)
- A is the disjoint union of the elementary events corresponding to its members
- A = $\{x_2, x_4, x_{22}\}$ A = $\{x_2\}$ $\{x_4\}$ $\{x_{22}\}$
- P(A) = P$\{x_2\}$+P$\{x_4\}$+P$\{x_{22}\}$ = $p_2 + p_4 + p_{22}$

Probability Axioms for finite spaces

- The ideas described thus far are the basis of the axioms of probability theory
- Probabilities are numbers assigned to events that satisfy the following rules
- Axiom I: P(A) ≥ 0 for all events A
- Axiom II: P(Ω) = 1
- Axiom III:If events A and B are disjoint, then P(A B) = P(A) + P(B)

Consequences of the Axioms

- Axiom II: P(Ω) = 1
- Axiom III: If events A and B are disjoint, then P(A B) = P(A) + P(B)
- Ω and are disjoint events
- P(Ω) = P(Ω) + P( ) (Axiom III)
  = 1 + P( ) (Axiom II)
- But, Ω = Ω
- Hence P(Ω) = P(Ω) = 1 P( ) = 0

More consequences of the Axioms

- Axiom II: P(Ω) = 1
- Axiom III: If events A and B are disjoint, then P(A B) = P(A) + P(B)
- A and A^c are disjoint events; A A^c = Ω
- 1 = P(Ω) = P(A A^c) = P(A) + P(A^c)
- P(A^c) = 1 – P(A); P(A) = 1 – P(A^c)
- Since P(A) ≥ 0 and P(A^c) ≥ 0 (Axiom I), we deduce that P(A) ≤ 1, P(A^c) ≤ 1

Summary

- We have discussed experiments with a finite number of outcomes
- The important concept of an event was introduced
- Probabilities were assigned to events
- The axioms of probability were introduced (for finite sample spaces)
- Some consequences of the axioms were derived

