**1.(a)** A and B are arbitrary events with 0 < P(A) < 1, 0 < P(B) < 1

| TRUE | FALSE | |
|---|---|---|
| ■ | □ | $P(A \quad B) = P(A^c \quad B^c) - P(A^c) - P(B^c) + 1$<br>$[P(A^c \quad B^c) - P(A^c) - P(B^c)] + 1 = - P(A^cB^c) + 1 = 1 - P((A \quad B)^c) = P(A \quad B)$ |
| □ | ■ | P(A÷B) > P(A)<br>The conditional probability P(A÷B) can be larger, smaller, or the same as P(A) |
| ■ | □ | P(A÷B) ≤ P(A)/P(B)<br>P(A÷B) = P(AB)/P(B) ≤ P(A)/P(B) since P(AB) ≤ P(A) |
| □ | ■ | $P(A÷B) + P(A÷B^c) = 1$ |
| □ | ■ | $P(A÷B) + P(A^c÷B^c) = 1$ |
| ■ | □ | $P(A÷B) + P(A^c÷B) = 1$<br>Conditional probabilities are a probability measure |
| ■ | □ | If P(A) = P(B), then P(A÷B) = P(B÷A)<br>Since P(A) = P(B), P(A÷B) = P(AB)/P(A) = P(AB)/P(B) = P(B÷A) |
| □ | ■ | If P(A÷B) = P(B÷A), then P(A) = P(B)<br>P(A÷B) = P(AB)/P(A) = P(AB)/P(B) = P(B÷A) P(A) = P(B) only if P(AB) ≠ 0 |
| ■ | □ | $P(A÷B)P(B) + P(A÷B^c)P(B^c) = P(A)$<br>This is just the theorem of total probability! |
| □ | ■ | P(A÷B) = P(B÷A)P(B)/P(A)<br>Bayes' formula gives P(A÷B) = P(B÷A)P(A)/P(B) |

**(b)** **X** and **Y** are arbitrary random variables with identical variance $\sigma^2$

| TRUE | FALSE | |
|---|---|---|
| □ | ■ | $E[\mathbf{X}^2] = E[\mathbf{Y}^2]$<br>True if $E[\mathbf{X}] = \pm E[\mathbf{Y}]$, but not in general. |
| □ | ■ | $\mathrm{var}(\mathbf{X} + \mathbf{Y}) = 2\sigma^2$<br>True if cov(**X**, **Y**) = 0, i.e. the RVs are uncorrelated, but not in general. |
| □ | ■ | $\mathrm{var}(\mathbf{X} - \mathbf{Y}) = 0$<br>True if P{**X** = **Y**} = 1, but not in general. |
| ■ | □ | $\mathrm{var}(\mathbf{X} + \mathbf{Y}) + \mathrm{var}(\mathbf{X} - \mathbf{Y}) = 4\sigma^2$<br>$\mathrm{var}(\mathbf{X} \pm \mathbf{Y}) = \mathrm{var}(\mathbf{X}) + \mathrm{var}(\mathbf{Y}) \pm 2\bullet\mathrm{cov}(\mathbf{X}, \mathbf{Y})$. Thus, $2\bullet\mathrm{cov}(\mathbf{X}, \mathbf{Y})$ cancels |
| ■ | □ | $\mathrm{var}(2\mathbf{X} + 3\mathbf{Y}) = \mathrm{var}(3\mathbf{X} + 2\mathbf{Y})$<br>Both variances equal $13\sigma^2 + 12\bullet\mathrm{cov}(\mathbf{X}, \mathbf{Y})$ |
| ■ | □ | $÷\mathrm{cov}(\mathbf{X}, \mathbf{Y})÷ \leq \sigma^2$<br>$÷\rho÷ = ÷\mathrm{cov}(\mathbf{X}, \mathbf{Y})/\sqrt{\mathrm{var}(\mathbf{X})\mathrm{var}(\mathbf{Y})}÷ = ÷\mathrm{cov}(\mathbf{X}, \mathbf{Y})÷/\sigma^2 \leq 1$ |
| ■ | □ | **X** + **Y** and **X** – **Y** are uncorrelated random variables<br>$\mathrm{cov}(\mathbf{X}+\mathbf{Y}, \mathbf{X}-\mathbf{Y}) = \mathrm{var}(\mathbf{X}) - \mathrm{var}(\mathbf{Y}) + \mathrm{cov}(\mathbf{X},\mathbf{Y}) - \mathrm{cov}(\mathbf{Y},\mathbf{X}) = 0$ |
| □ | ■ | **X** + **Y** and **X** – **Y** are independent random variables<br>Uncorrelated random variables are not necessarily independent |

**3.** There are 3 red and 3 black balls in the urn.

**(a)** $(R_1 \quad R_2) \cap (B_1 \quad B_2) = (R_1 \cap B_2) \quad (B_1 \cap R_2)$ since $R_1 \cap B_1 = R_2 \cap B_2 = \quad$ .

Hence, $P\{(R_1 \quad R_2) \cap (B_1 \quad B_2)\} = P(R_1)P(B_2 \mid R_1) + P(B_1)P(R_2 \mid B_1) = \frac{3}{6} \times \frac{3}{5} + \frac{3}{6} \times \frac{3}{5} = \frac{3}{5}$.

$P\{B_1 \mid (R_1 \quad R_2)\} = \dfrac{P\{B_1 \cap (R_1 \quad R_2)\}}{P\{R_1 \quad R_2\}} = \dfrac{P\{B_1 \cap R_2\}}{P\{R_1 \quad R_2\}}$. We have already found $P\{B_1 \cap R_2\} = \frac{3}{6} \times \frac{3}{5}$.

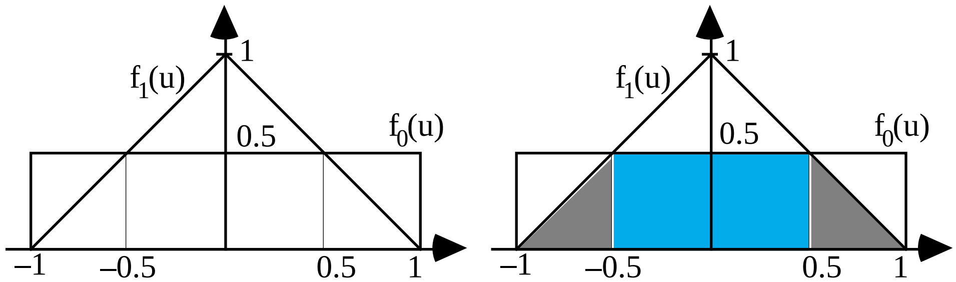
Since $P\{R_1 \quad R_2\} = 1 - P\{B_1 \cap B_2\} = 1 - \frac{3}{6} \times \frac{2}{5} = \frac{24}{30} = \frac{4}{5}$, we get that $P\{B_1 \mid (R_1 \quad R_2)\} = \frac{3}{8}$.

**(b)** $R_1$ and $R_2$ are neither disjoint nor independent. However,

$P(R_2) = P(R_2 \mid B_1)P(B_1) + P(R_2 \mid R_1)P(R_1) = \frac{3}{5} \times \frac{3}{6} + \frac{2}{5} \times \frac{3}{6} = \frac{1}{2} = P(R_1)$.

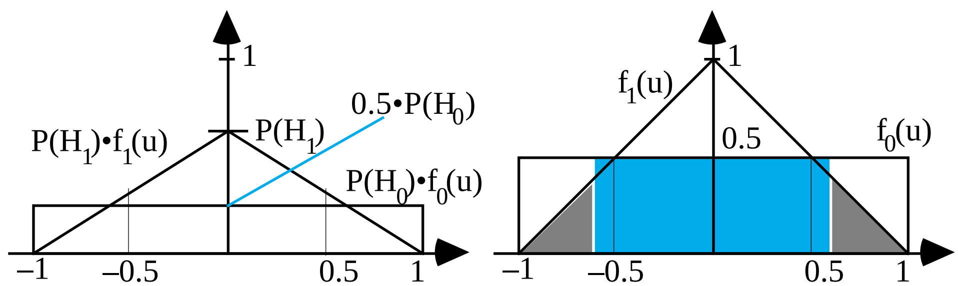
**4.** $E[\mathbf{Y}] = 2 = (b+a)/2$. $\text{var}(\mathbf{Y}) = 3 = (b-a)^2/12$ giving that $b-a = 6$. Hence, $b = 5$, $a = -1$, and $P\{\mathbf{Y} \le 1\} = 1/3$.

**5.** The pdfs under the two hypotheses are as shown below.




**(a)** The maximum-likelihood decision rule chooses $H_1$ if the observed value of $\theta$ is in the interval $(-0.5, 0.5)$ and $H_0$ if the observed value of $\theta$ is outside this interval. Hence, $P_{FA}(ML) = 1/2$ and $P_{MD}(ML) = 1/4$.

**(b)** If $P(H_0) \ne P(H_1)$, the Bayes' rule compares $P(H_0) \cdot f_0(u) = 0.5 \cdot P(H_0)$ and $P(H_1) \cdot f_1(u) = (1-|u|) \cdot P(H_1)$. Since $P(H_0) < 0.5$, $P(H_1) > 0.5$, we see that $1 - |u|$ must exceed $0.5 \cdot P(H_0)/P(H_1)$ which is less than 0.5. Hence, the Bayes rule decides in favor of $H_1$ if $|\theta| < 1 - 0.5 \cdot P(H_0)/P(H_1)$ where the RHS is larger than 0.5. The situation is as shown below, and we conclude that $P_{FA}(\text{Bayes}) > P_{FA}(ML)$ and $P_{MD}(\text{Bayes}) < P_{MD}(ML)$. Thus, only ` is a true statement. This makes sense intuitively. Since $P(H_1) > P(H_0)$, the Bayes rule attempts to reduce the chances of error (missed detection) when the more likely hypothesis is true, while accepting a somewhat larger chance of error (false alarm) when the less likely hypothesis is true.




**(c)** From the left-hand diagram above, we see that the Bayes decision rule will always choose $H_0$ if $0.5 \cdot P(H_0)$ is larger than $P(H_1) = 1 - P(H_0)$, i.e. if $P(H_0) > 2/3$.

**6.(a)** $P\{2\mathbf{Y} < \mathbf{X}\} = P\{\mathbf{Y} < \mathbf{X}/2\} = \int_{u=0}^{1}\int_{v=0}^{u/2} u + v \, dv\, du = \int_{u=0}^{1} u^2/2 + u^2/8 \, du = \frac{1}{6} + \frac{1}{24} = \frac{5}{24}$

**(b)** For $0 \le u \le 1$, $f_{\mathbf{X}}(u) = \int_0^1 u + v \, dv = u + 1/2$. For all other values of u, $f_{\mathbf{X}}(u) = 0$.

Sanity check: $f_{\mathbf{X}}(u) \ge 0$ and $\int_0^1 u + 1/2 \, du = 1$

**(c)** $E[\mathbf{Y}] = \int_0^1 \int_0^1 v \cdot (u + v)\, du\, dv = \int_0^1 v/2 + v^2 \, dv = \frac{1}{4} + \frac{1}{3} = \frac{7}{12} = E[\mathbf{X}]$. Why?

**(d)** $E[\mathbf{XY}] = \int_0^1 \int_0^1 uv \cdot (u + v)\,du\,dv = \int_0^1 v/3 + v^2/2\,dv = \frac{1}{6} + \frac{1}{6} = \frac{1}{3}$.

Hence, $\mathrm{cov}(\mathbf{X}, \mathbf{Y}) = E[\mathbf{XY}] - E[\mathbf{X}]E[\mathbf{Y}] = \frac{-1}{144}$.

**(e)** Since $f_{\mathbf{X}}(0.4) = u + 0.5 = 0.9$ from part (a), $f_{\mathbf{Y}|\mathbf{X}}(v|0.4)$, the conditional pdf of $\mathbf{Y}$ given that $\mathbf{X} = 0.4$, is given by

$$f_{\mathbf{Y}|\mathbf{X}}(v|0.4) = \frac{f_{\mathbf{X},\mathbf{Y}}(0.4,v)}{f_{\mathbf{X}}(0.4)} = \begin{cases} \frac{10}{9}\cdot(0.4 + v) & \text{if } 0 \le v \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

**(f)** $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$ can take on values in the range [0, 2]. For $0 \le \alpha \le 1$, $F_{\mathbf{Z}}(\alpha) = P\{\mathbf{Z} \le \alpha\} = P\{\mathbf{X} + \mathbf{Y} \le \alpha\}$

$$= \int_{v=0}^{\alpha} \int_{u=0}^{\alpha - v} u + v\,du\,dv = \int_{v=0}^{\alpha} (\alpha - v)^2/2 + v(\alpha - v)\,dv = -(\alpha - v)^3/6 + \alpha v^2/2 - v^3/3 \Big|_0^{\alpha} = \alpha^3/3.$$

For $1 \le \alpha \le 2$, $1 - F_{\mathbf{Z}}(\alpha) = P\{\mathbf{Z} > \alpha\} = P\{\mathbf{X} + \mathbf{Y} > \alpha\} = \int_{v=\alpha-1}^{1} \int_{u=\alpha-v}^{1} u + v\,du\,dv$

$$= \int_{v=\alpha-1}^{1} v + 1/2 - (\alpha - v)^2/2 - v(\alpha - v)\,dv = v^2/2 + v/2 + (\alpha - v)^3/6 - \alpha v^2/2 + v^3/3 \Big|_{\alpha-1}^{1}$$

$= \alpha^3/3 - \alpha^2 + 4/3$, giving $f_{\mathbf{Z}}(\alpha) = \begin{cases} \alpha^2, & 0 \le \alpha \le 1, \\ \alpha(2-\alpha), & 1 \le \alpha \le 2, \\ 0, & \text{otherwise.} \end{cases}$ Sanity check: $F_{\mathbf{Z}}(\alpha) = \alpha^3/3$ increases from 0 to 1/3 as $\alpha$ increases from 0 to 1, while $F_{\mathbf{Z}}(\alpha) = 1 - (\alpha^3/3 - \alpha^2 + 4/3)$ has value 1/3 at $\alpha = 1$ and increases to 1 at $\alpha = 2$.

**7.(a)** $\mathbf{X}$ is a binomial random variable with parameters (25600, 0.5).

**(b)** $E[\mathbf{X}] = 25600 \cdot 0.5 = 12800$. $\mathrm{var}(\mathbf{X}) = 25600 \cdot 0.5 \cdot 0.5 = 6400$.

**(c)** Since $\mathbf{X} + \mathbf{Y} = 25600$, $\mathbf{X}$ and $\mathbf{Y}$ cannot be independent random variables. Knowing one tells is the other.

**(d)** $\mathbf{X}$ and $\mathbf{Y}$ are either both even or both odd. Hence, $\mathbf{X} - \mathbf{Y}$ cannot equal the lead of B over A; one or the other of A and B must win.

**(e)** $\mathbf{Z} = \mathbf{X} - \mathbf{Y} = 2\mathbf{X} - 25600$ has mean $E[\mathbf{Z}] = 2 \cdot E[\mathbf{X}] - 25600 = 0$ and variance $\mathrm{var}(\mathbf{Z}) = 4 \cdot \mathrm{var}(\mathbf{X}) = 25600$.

**(f)** According to the Central Limit Theorem, the CDF of $\mathbf{Z}$ can be approximated by the CDF of a N(0,25600) random variable. Thus, $P(\text{B wins}) = P\{\mathbf{Z} \le 536\} \approx \Phi(536/\sqrt{25600}) = \Phi(536/160) = \Phi(3.35) \approx 0.9996$

**(g)** Now, $\mathbf{X}$ is binomial with parameters (25600,p) and thus $\mathbf{Z} = \mathbf{X} - \mathbf{Y} = 2\mathbf{X} - 25600$ now has mean $E[\mathbf{Z}] = 2 \cdot E[\mathbf{X}] - 25600 = 2 \cdot 25600 \cdot p - 25600 = 25600 \cdot (2p-1)$ and variance $4 \cdot \mathrm{var}(\mathbf{X}) = 25600 \cdot p \cdot (1-p)$.

In order for $P(\text{A wins}) = P\{\mathbf{Z} \ge 538\} \approx 1 - \Phi\left(\frac{538 - E[\mathbf{Z}]}{\sqrt{\mathrm{var}(\mathbf{Z})}}\right) = 1 - \Phi\left(\frac{538 - 25600 \cdot (2p-1)}{\sqrt{\mathrm{var}(\mathbf{Z})}}\right)$ to be 0.5 or more, the argument of $\Phi$ must be at most zero, i.e. $538 \le 25600 \cdot (2p-1)$, that is,

$p \ge \frac{538 + 25600}{2 \cdot 25600} = 0.5 + \frac{538}{51200} = 0.5105\ldots$ Note that it is not necessary to compute $\mathrm{var}(\mathbf{Z})$ at all!