ECE 313 FINAL EXAMINATION
Monday December 11, 2000
Three hours

1. **(36 points)** Check the appropriate box for each of the statements below. No justification is required, but, in order to discourage guessing, your score will be reduced by 2 points for each wrong answer (You get +2 points for each right answer; 0 for no answer).

(a) Which of the following statements are true for **all** events A and B such that 0 < P(A) < 1 and 0 < P(B) < 1 ?

TRUE	FALSE	
n	n	$P(A \quad B) = P(A^c \quad B^c) – P(A^c) – P(B^c) + 1$
n	n	P(A÷B) > P(A)
n	n	P(A÷B) ≤ P(A)/P(B)
n	n	$P(A÷B) + P(A÷B^c) = 1$
n	n	$P(A÷B) + P(A^c÷B^c) = 1$
n	n	$P(A÷B) + P(A^c÷B) = 1$
n	n	If P(A) = P(B), then P(A÷B) = P(B÷A)
n	n	If P(A÷B) = P(B÷A), then P(A) = P(B)
n	n	$P(A÷B)P(B) + P(A÷B^c)P(B^c) = P(A)$
n	n	P(A÷B) = P(B÷A)P(B)/P(A)

(b) Which of the following statements are true for **all** random variables **X** and **Y** with identical finite variance σ^2 ?

TRUE	FALSE	
n	n	$E[\mathbf{X}^2] = E[\mathbf{Y}^2]$
n	n	$var(\mathbf{X} + \mathbf{Y}) = 2\sigma^2$
n	n	$var(\mathbf{X} – \mathbf{Y}) = 0$
n	n	$var(\mathbf{X} + \mathbf{Y}) + var(\mathbf{X} – \mathbf{Y}) = 4\sigma^2$
n	n	$var(2\mathbf{X} + 3\mathbf{Y}) = var(3\mathbf{X} + 2\mathbf{Y})$
n	n	$÷cov(\mathbf{X}, \mathbf{Y})÷ \leq \sigma^2$
n	n	**X** + **Y** and **X** – **Y** are uncorrelated random variables
n	n	**X** + **Y** and **X** – **Y** are independent random variables

2. **(24 points)** Check **one box** for each of the following questions. No justification is required, but, your score will be reduced by 2 points for each wrong answer (You get +8 points for each right answer; 0 for no answer).

(a) Which of the following four statements are true for all events A and B with probabilities P(A) and P(B) such that 0 < P(A) < 1, 0 < P(B) < 1?

‹ P(A ∩ B) ≤ min{P(A), P(B)} ˋ P(A B) ≥ [P(A) + P(B)]/2

´ P(A ∩ B) ≥ P(A) + P(B) – 1. ˆ P(A ∩ B) ≤ P(A÷B)

- n **Only** ‹ is a true statement.
- n **Only** ‹ and ˋ are true statements.
- n **Only** ‹ , ˋ , and ˆ are true statements.
- n **All four** are true statements.
- n **None of the four** are true statements.

(b) Which of the following four statements are **NOT** properties of **all** CDFs?

‹ $P\{\mathbf{X} > b\} = 1 – F_{\mathbf{X}}(b)$. ˋ If $F_{\mathbf{X}}(a) < F_{\mathbf{X}}(b)$, then a < b.

´ If a < b, then $F_{\mathbf{X}}(a) < F_{\mathbf{X}}(b)$. ˆ $F_{\mathbf{X}}(u) = 0.5$ for some u, $–\infty < u < \infty$.

- n **Only** ` is not a property of CDFs.
- n **Only** ` and ´ are not properties of CDFs.
- n **Only** ´ is not a property of CDFs.
- n **Only** ´ and ˆ are not properties of CDFs.
- n You blew it, Professor! **All four** are properties of CDFs.

(c) Which of the following four statements are properties of **all** pdfs?

‹ $f_X(u) \le 1$ for all u, $-\infty < u < \infty$. ` $\lim_{u\to -\infty} f_X(u) = 0$

´ $\lim_{u\to \infty} f_X(u) = 1$ ˆ $P\{a < \mathbf{X} < b\} = P\{a \le \mathbf{X} \le b\}$

- n **Only** ‹ , ` , and ˆ are properties of pdfs.
- n **Only** ` and ˆ are properties of pdfs.
- n **Only** ` , ´ and ˆ are properties of pdfs.
- n **All four** are properties of pdfs.
- n None of the above. Only the following are properties of pdfs:________

3. **(18 points)** An urn contains 3 red and 3 black balls. Two balls are drawn in succession from the urn. The first ball is not replaced in the urn before the second is drawn.

For i = 1, 2, let R_i denote the event that the i-th ball drawn is red

and $B_i = R_i^c$ denote the event that the i-th ball drawn is black.

(a) **(12 points)** Find $P\{(R_1 \quad R_2) \cap (B_1 \quad B_2)\}$ and $P\{B_1 \mid (R_1 \quad R_2)\}$.

(b) **(6 points)** Mark one box for each of the three statements. You do not need to justify your answers but your total score on Problem 3 will be reduced by 2 points for each wrong answer (You get +2 points for each right answer; 0 for no answer)

(i)	R_1 and R_2 are disjoint (i.e. mutually exclusive) events	n TRUE	n FALSE
(ii)	R_1 and R_2 are independent events	n TRUE	n FALSE
(iii)	$P(R_1) = P(R_2)$	n TRUE	n FALSE

4. **(15 points)** A continuous random variable **Y** is uniformly distributed on the interval [a, b]. It is known that **Y** has mean 2 and *standard deviation* $\sqrt{3}$. Find $P\{\mathbf{Y} \le 1\}$. If the answer cannot be determined from the given information, check the box and leave the right hand portion of the answer area blank.

5. **(30 points)** A phase-lock loop (PLL) with noiseless input $\cos(\omega t)$ produces an output $\sin(\omega t)$. However, since noise is always present, the PLL output is usually modeled as $\sin(\omega t + \pi\Theta)$ where Θ is a continuous random variable taking on values in the interval [–1, 1]. If the input to the PLL is the signal $\cos(\omega t)$ *plus* noise (this is **hypothesis H_1**), the pdf of Θ is given by

$$f_1(u) = \begin{cases} 1 - |u|, & -1 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

On the other hand, if the signal is absent and the input to the PLL is just noise, (this is **hypothesis H_0**), then Θ is uniformly distributed on [–1, 1]. The receiver observes the value of Θ and must decide whether or not the signal $\cos(\omega t)$ is present in the PLL input.

(a) **(14 points)** Find $P_{FA}(ML)$, the false-alarm probability, and $P_{MD}(ML)$, the missed-detection probability, of the maximum-likelihood decision rule.

(b) **(8 points)** Now suppose that $P(H_0) < 0.5$ and $P(H_1) = 1 - P(H_0) > 0.5$, and consider $P_{FA}(Bayes)$, the false-alarm probability, and $P_{MD}(Bayes)$, the missed-detection probability, of the Bayes decision rule (also called the minimum-error-probability or maximum *a posteriori* probability decision rule.) Which of the following statements are true? You need not justify your answer, but, your total score on Problem 5 will be reduced by 2 points for a wrong answer (You get +8 points for right answer; 0 for no answer)

‹ P_{FA}(Bayes) < P_{FA}(ML) ` P_{MD}(Bayes) < P_{MD}(ML)

- n Only ‹ is a true statement
- n Both ‹ and ` are true statements
- n Only ` is a true statement
- n Neither ‹ nor ` is a true statement

(c) **(8 points)** For what value(s) of $P(H_0)$ does the Bayes decision rule always choose H_0 regardless of the observed value of Θ ?

6. **(54 points)** **X** and **Y** are jointly continuous random variables with joint probability density function (pdf) given by $f_{X,Y}(u, v) = \begin{cases} u + v, & 0 \le u \le 1, 0 \le v \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

(a) **(9 points)** What is the value of P{2**Y** < **X**}?

(b) **(9 points)** Find the marginal pdf of **X**. To obtain full credit, you must specify the value of $f_X(u)$ for all u, $-\infty < u < \infty$.

(c) **(9 points)** What is the expected value of **Y**?

(d) **(9 points)** What is the covariance of **X** and **Y**?

(e) **(9 points)** Find the conditional pdf of **Y** given that **X** = 0.4. To obtain full credit, you must specify the value of $f_{Y|X}(v|0.4)$ for all v, $-\infty < v < \infty$.

(f) **(9 points)** Let **Z** = **X** + **Y**. Find $f_Z(\alpha)$, the probability density function of **Z**. To obtain full credit, you must specify the value of $f_Z(\alpha)$ for all α, $-\infty < \alpha < \infty$.

7. **(48 points)** Voters in the State of Pallida, one of the fifty States comprising the Utopian States of America, have voted for one of two candidates A and B. The "certified count" of these votes shows B leading A by 537 votes. However, there are 25,600 ballots that are in dispute, and the totals of these votes have not been included in the "certified count". This may change following the resolution of various court cases.

Suppose that each of the 25,600 ballots is *equally likely* to be marked *correctly* for either A or B, with each ballot marking being independent of all the others. Thus, there are no double punches, blank ballots, hanging or pregnant chads, late or missing postmarks, etc. Let **X** and **Y** respectively denote the numbers of votes for A and B in these 25,600 ballots.

(a) **(4 points)** What kind of random variable is **X**? Check one box and state the parameters .

- n Bernoulli random variable with parameter p =
- n Poisson random variable with parameter λ =
- n Geometric random variable with parameter p =
- n Binomial random variable with parameters (n, p) = ( )
- n Negative binomial RV with parameters (r, p) = ( )
- n None of the above: the pmf of **X** is given by $p_X(k)$ =

(b) **(4 points)** What is the mean of **X**? What is the variance of **X**?
If you checked "None of the above" in part (a), then you *must show* how you computed E[**X**] and var(**X**) on the back of the previous page. If you checked any of the other boxes, no work need be shown.

(c) **(2 points)** Are **X** and **Y** independent random variables? Explain your answer in 30 or fewer words.

- n Yes, **X** and **Y** are independent random variables
- n No, **X** and **Y** are not independent random variables

(d) **(2 points)** If the 25,600 ballots are included in the final count, is it possible that both A and B will end up with exactly the same number of votes? No explanation is required.

- n Yes, A and B can end up with identical numbers of votes, resulting in a tie.
- n No, A and B cannot end up with identical numbers of votes; one or the other must win.

(e) **(12 points)** Assuming that the 25,600 disputed ballots are included in the final count, what is the mean and variance of **Z** = **X** – **Y**, the "net gain" of candidate A in the polls?

(f) **(12 points)** Assuming that the 25,600 ballots are included in the final count, use the Central Limit Theorem to estimate the probability that B wins the election in Pallida.
Hint: $\sqrt{25{,}600} = 160$

Now suppose that the 25,600 disputed ballots are from areas where A is more likely to get votes than B. Thus, each ballot is likely to be marked for A with probability $p > 0.5$ and for B with probability $1 - p < 0.5$ where $0.5 < p \leq 1$. As before, each ballot marking is independent of all others. Note that if $p = 1$, A gets all 25,600 votes and wins (if the Court allows these ballots to be included in the final count.) For somewhat smaller values of p, there is still a "reasonable probability" that A will overcome B's lead in the vote count.

(g) **(12 points)** Assuming that the Court allows all the 25,600 disputed ballots to be included in the final count, use the Central Limit Theorem to estimate the *minimum* value of p such that the value of P{A wins the election in Pallida} is *at least* 0.5.