Assigned: Wednesday, December 1, 1999
Due: Wednesday, December 8, 1999
Reading: Chapter 7 (Sections 1-3 only), Chapter 8
Problems:

1. If X is $N(0,\sigma^2)$, then $X^2$ has gamma pdf with parameter $(1/2,1/2\sigma^2)$. We did the case for $\sigma^2 = 1$ in class, and the general case is very similar. Now, suppose that X, Y, and Z are independent $N(0,\sigma^2)$ random variables. Then $X^2$, $Y^2$, and $Z^2$ are independent gamma random variables with parameter $(1/2,1/2\sigma^2)$.

(a) Use the comment immediately following the proof of Proposition 3.1 (pp. 266-267) of Ross to *state* what the *type* of pdf of $W = X^2 + Y^2 + Z^2$ is, and write down *explicitly* the exact pdf. What *is* the numerical value of $f_W(5)$ if $\sigma^2 = 4$?

(b) Use LOTUS to prove that $E[W] = 3\sigma^2$. If you actually evaluated an integral to get this answer, shame on you!

(c) In a physical application, X, Y, and Z represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m. Thus, $H = (1/2)mW$ is the kinetic energy of the particle, and an important axiom of statistical mechanics asserts that the average kinetic energy is $E[H] = E[(1/2)mW] = (1/2)mE[W] = (3/2)m\sigma^2 = (3/2)kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in °K. (Note that the average energy is (1/2)kT per dimension.) Show that the kinetic energy H has the Maxwell-Boltzmann pdf $f_H(\beta) = \frac{2}{\sqrt{\pi}}(kT)^{-3/2}\sqrt{\beta}\,\exp(-\beta/kT),\ \beta > 0.$

(d) $V = \sqrt{W} = \sqrt{X^2 + Y^2 + Z^2}$ is the "speed" of the molecule. Show that the pdf of V is $f_V(\gamma) = \frac{4}{\sqrt{\pi}}\left(\frac{m}{2kT}\right)^{3/2}\gamma^2 \exp\left(-\frac{m\gamma^2}{2kT}\right)$, $\gamma > 0$ cf. Theoretical Exercise 1, p. 237 of the text.

(e) What is the average speed of the molecule?

2. Except for the trivial case when all the probability mass is at $\mu = E[X]$, there is probability mass both to the left and right of $\mu$; in particular, there is an $\omega \in \Omega$ such that $X(\omega) < \mu$. Is it also true that if $(E[X], E[Y]) = (\mu_1, \mu_2)$, then there is an $\omega \in \Omega$ such that $X(\omega) < \mu_1$ and $Y(\omega) < \mu_2$? If you believe the result is true, prove it. Otherwise, give a counterexample to show that it is false.

3. In economics, an isoquant is a curve which shows all the possible combinations of inputs which yield the same output. Suppose that the output, which is ball bearings, is a function of capital and labor, which are inputs. You own two plants, Plant 1 and Plant 2. The number of ball bearings produced by plant i is the r.v. $B_i$. The labor input (man hours) for Plant i is $L_i$ and the capital input ($) for Plant i is the r.v. $C_i$. The labor and capital are independent and continuous random variables. The plants are set up identically so that $B_i = L_i C_i$. The costs are $D_i = L_i + 2 C_i$

(your credit rating is lousy and you have to borrow form a loan shark). The profits are the r.v.s $P_i = B_i - D_i$. The total profit is the r.v. $P = P_1 + P_2$.

(a) Draw the isoquants for Plant i for $B_i$ = 100, 200, 500 – that is plot the curves $B_i$ = 100, 200, 500 against $L_i$ and $C_i$.

(b) The manager of plant 1, M1, manages his plant in the following way. He asks for $l_1$ units of labor from the union and gets an amount which is uniformly distributed between $l_1/2$ and $3l_1/2$ (he has to take whatever they give him, even if it is more than he wanted). Then, depending on how much labor he gets, he asks the bank for $c_1$ amount of money and gets an amount which is uniformly distributed between $c_1$ and $c_1/2$ (again, he has to take whatever they give him). He has the following constraints: $4 < l_1 < 1000$, $4 < c_1 < 1000$. Find the E[ $B_1$] in terms of $l_1$ and $c_1$.

(c) Find E[$P_1$].

(d) What value of $c_1$ maximizes E[$P_1$]?

(e) Find the value(s) of $l_1$ which maximizes E[$P_1$].

(d) Find the pdf of $P_1$ in terms of $l_1$.

(e) The manager of plant 2, M2, manages her plant differently. She first asks the bank for $c_2$ amount of money and gets an amount which is uniformly distributed between $c_2$ and $c_2/2$ (she has to take whatever they give her). Then, depending on how much capital she gets, she asks for $l_2$ units of labor from the union and gets an amount which is uniformly distributed between $l_2/2$ and $3l_2/2$ (she has to take whatever they give her). She has the following constraints: $4 < l_2 < 1000$, $4 < c_2 < 1000$. Find the expected value of $B_2$ in terms of $l_2$ and $c_2$.

(f) Find the value(s) of $c_2$ which E[$P_2$].

(h) Which of the two managers is doing a better job at maximizing expected profits?

(i) Your (now) only manager now requests a single amount of capital for the two plants. He/she can distribute capital between the plants as he/she pleases, but not labor (the two plants are far away). The labor for the two plants is independent. Suppose that the manager follows whatever policy (capital first, or labor first) he/she followed originally. Find the expectation of P.

(j) Find the joint pdf of $P_1$ and $P_2$.

(k) You decide to consolidate operations and get rid of M1 and keep only M2 to be in charge of both plants. She still gets money first and labor second. She can either get all the capital as one sum C and distribute it equally between the two plants (each plant gets C/2), or get a separate capital for each plant as before (the money given to the two plants are independent). If she opts for the first option, then for c amount of money and gets an amount which is uniformly distributed between c and c/2 (she has to take whatever they give her). The constraint is $8 < c < 2000$. She still gets the labor in each plant separately. For the first option, C, $L_1$ and $L_2$ are mutually independent. For the second option, $L_1$, $L_2$, $C_1$ and $C_2$ are mutually independent.

4. The Sirrah Poll wishes to assess the popularity of Knute Gingpoor. To this end, a random sample of n persons is asked for opinions, with the opinion of the i–th person being denoted by $\mathbf{X}_i$ where $\mathbf{X}_i = 1$ if the person supports Knute, and $\mathbf{X}_i = 0$

if the person does not support Knute. The Sirrah Poll treats the $\mathbf{X}_i$'s as *independent* random variables with $P\{\mathbf{X}_i = 1\} = p$ for all i, and estimates p as $(\Re \mathbf{X}_i)/n$. The Poll wishes to be *fairly sure* that its estimate of p has a margin of error of 2% or less. The Poll thus wants to have the following inequality hold:

$P\{|(\Re \mathbf{X}_i)/n - p| > 0.02\}$ _ 0.05.

In other words, with high probability (0.95), the *estimate of p* differs from the *actual value of p* by at most 0.02 (2%). That evening, the networks announce that the Sirrah Poll had found that Knute Gingpoor has a popularity rating of $100[(\Re \mathbf{X}_i)/n]$% and that the margin of error of the poll is ±2%.

(a) Suppose that p = 0.1. Use the weak law of large numbers to find N such that for all n _ N, the above inequality is guaranteed to hold. Thus, the Poll should have used a random sample of size N or more. What if p = 0.2 ? 0.3 ? . . .

(b) Express N as a function of p. What is the maximum value of this function?

(c) The Sirrah Poll naturally wishes to minimize the number of persons surveyed in order to minimize the cost. However, the value of p is unknown. How many voters should the Poll survey so that, *regardless of the value of p,* the above inequality will be satisfied?

5.(a) A fair die is rolled. Let X denote the outcome (i.e. the number showing). What is the mean and variance of X?

(b) The die is rolled 1000 times and the 1000 outcomes are added together. The result is denoted by Y. What is the minimum value of Y? What is the maximum value?

(c) Estimate $P\{Y < 3500\}$.