Assigned: **Monday**, November 22, 1999
Due: **Wednesday**, December 1, 1999
Reading: Ross, Chapter 7, Sections 7.1-7.3, 7.4.1, 7.4.3, 7.5, and 7.7.1, Chapter 8
Noncredit Exercises: pp. 372-384: 1, 3-5, 7, 22, 26, 29, 34, 35, 44, 46, 47;
pp. 384-392: 1, 2, 19, 22, 41.

Problems:

1. Consider the random point $(\mathbf{X},\mathbf{Y})$ of Problem 2 of Problem Set #12.

(a) Compute $E[\mathbf{X}]$ and $\mathrm{var}(\mathbf{X})$.

(b) Explain why the random variable $\mathbf{Y}$ has the same mean and variance as $\mathbf{X}$.

(c) Compute $E[\mathbf{XY}]$ and hence find $\mathrm{cov}(\mathbf{X},\mathbf{Y})$.

(d) GLOTUS tells us that $E[g(\mathbf{X},\mathbf{Y})] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v)\cdot f_{\mathbf{X},\mathbf{Y}}(u,v)\,du\,dv$. Now, consider the function $g(\mathbf{X},\mathbf{Y}) = \min\{\mathbf{X},\mathbf{Y}\} = \begin{cases} \mathbf{X}, & \text{if } \mathbf{X} < \mathbf{Y}, \\ \mathbf{Y}, & \text{if } \mathbf{Y} \le \mathbf{X}. \end{cases}$

Use GLOTUS to find $E[g(\mathbf{X},\mathbf{Y})] = E[\min\{\mathbf{X},\mathbf{Y}\}]$ by showing that the glotal integrand $g(u,v)\cdot f_{\mathbf{X},\mathbf{Y}}(u,v)$can be expressed as

$u\cdot f_{\mathbf{X},\mathbf{Y}}(u,v)$ for all points (u,v) in the plane for which $u < v$

and as $v\cdot f_{\mathbf{X},\mathbf{Y}}(u,v)$ for all points (u,v) in the plane for which $v \le u$.

Thus, the glotal integral can be expressed as the sum of integrals over two disjoint regions (these are divided by the line of slope 1 through the origin) of the plane.

Reminder: In case you got lost in the above verbiage, you are to find $E[\min\{\mathbf{X},\mathbf{Y}\}]$.

(e) Repeat part (d) to find $E[h(\mathbf{X},\mathbf{Y})] = E[\max\{\mathbf{X},\mathbf{Y}\}]$.

(f) Compare the numerical values that you obtained in parts (d) and (e), and state whether or not $E[\max\{\mathbf{X},\mathbf{Y}\}]$ is larger than $E[\min\{\mathbf{X},\mathbf{Y}\}]$.
Should $E[\max\{\mathbf{X},\mathbf{Y}\}]$ exceed $E[\min\{\mathbf{X},\mathbf{Y}\}]$ (even if, according to your computed values, it does not in this instance)?

(g) Since $\min\{\mathbf{X},\mathbf{Y}\} + \max\{\mathbf{X},\mathbf{Y}\} = \mathbf{X} + \mathbf{Y}$, the following equation

$$E[\min\{\mathbf{X},\mathbf{Y}\}] + E[\max\{\mathbf{X},\mathbf{Y}\}] = E[\mathbf{X}] + E[\mathbf{Y}]$$

should hold (as explained in Problem 7 of Problem Set #12.) ***Is*** the above equation satisfied by the numerical values that you have obtained?

(h) Show that the conditional pdf of $\mathbf{Y}$ given $\mathbf{X}$ is uniform on [0.5, 1] if $\mathbf{X} \le 0.5$, and is uniform on [0, 1] if $\mathbf{X} > 0.5$.

(i) The **best** (least mean-square error) estimate of $\mathbf{Y}$ given that $\mathbf{X}$ is known to have value α is the mean of the conditional pdf of $\mathbf{Y}$ given $\mathbf{X} = \alpha$. Thus, it follows from part (h) that if $\mathbf{X}$ has value $\alpha \le 0.5$, then $\hat{\mathbf{Y}}$, the best estimate of $\mathbf{Y}$, is 0.75 while if $\mathbf{X}$ has value $\alpha > 0.5$, then $\hat{\mathbf{Y}} = 0.5$. Now, the **best linear** (least mean-square error) estimate of $\mathbf{Y}$ (given that $\mathbf{X}$ is known to have value α) is $\tilde{\mathbf{Y}} = a + b\alpha$ where a and b are given in Eq. (5.4) on page 353 of Ross. Compute a and b, and draw a graph showing the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ as functions of α. (Remember that $0 \le \alpha \le 1$). For what value(s) of α are the two estimates the same?

(j) Since the estimates $\hat{\mathbf{Y}}$ and $\tilde{\mathbf{Y}}$ depend on the value of $\mathbf{X}$, they really are *functions* of $\mathbf{X}$, that is, they are *random variables* that can be expressed as $\hat{\mathbf{Y}} = \begin{cases} 0.75, & 0 \le \mathbf{X} \le 0.5, \\ 0.5, & 0.5 < \mathbf{X} \le 1 \end{cases}$ and $\tilde{\mathbf{Y}} = a + b\mathbf{X}$. What are the average and the mean-square errors of each estimate? That is, what are the values of $E[(\mathbf{Y} - \hat{\mathbf{Y}})]$, $E[(\mathbf{Y} - \tilde{\mathbf{Y}})]$, $E[(\mathbf{Y} - \hat{\mathbf{Y}})^2]$, and $E[(\mathbf{Y} - \tilde{\mathbf{Y}})^2]$?

(k) There is no part (k).

2. Let the random variables **X** and **Y** be independent and uniformly distributed on (0,1). Find E(|**X**–**Y**|) and Var(**X**–**Y**).

3. Let E[**X**] = 1, E[**Y**] = 4, var(**X**) = 4, var(**Y**) = 9, and $\rho_{X,Y} = 0.1$

(a) If **Z** = 2(**X**+**Y**)(**X**–**Y**), what is E[**Z**]?

(b) If **T** = 2**X**+**Y** and **U** = 2**X**–**Y**, what is cov(**T**, **U**)?

(c) If **W** = 3**X** + **Y** + 2, find E[**W**] and var(**W**).

(d) If **X** and **Y** are jointly Gaussian random variables, and **W** is as defined in (c), what is P{**W** > 0} ?

4. This problem has three independent parts. Do not apply the numbers from one part to the others.

(a) If var(**X** + **Y**) = 36 and var(**X** – **Y**) = 64, what is cov(**X**, **Y**) ? If you are also told that var(**X**) = 3•var(**Y**), what is $\rho_{X,Y}$?

(b) If var(**X** + **Y**) = var(**X** – **Y**), are **X** and **Y** uncorrelated ?

(c) If var(**X**) = var(**Y**), are **X** and **Y** uncorrelated ?