University of Illinois | **Problem Set #12** | **ECE 313**

of Illinois | **Page 1 of 2** | **Fall 1999**

Assigned: **Monday**, November 15, 1999
Due: **Monday**, November 22, 1999
Reading: Ross, Chapter 6 (except Sections 6.6 and 6.8), Chapter 7, Sections 1-3
Noncredit Exercises: pp. 293-300: 16, 33, 40-43, 51-54.
Problems:

1. Let ($\mathbf{X}$, $\mathbf{Y}$) have joint pdf $f_{\mathbf{X},\mathbf{Y}}(u, v) = \begin{cases} C\sqrt{1-u^2-v^2}, & u^2+v^2 < 1, \\ 0, & \text{elsewhere.} \end{cases}$

(a) What is the value of C?

(b) Find $P\{\mathbf{X}^2+\mathbf{Y}^2 < 0.25\}$.

(c) What is $f_{\mathbf{X}|\mathbf{Y}}(u|0.6)$, the conditional pdf of $\mathbf{X}$ given that $\mathbf{Y} = 0.6$?.

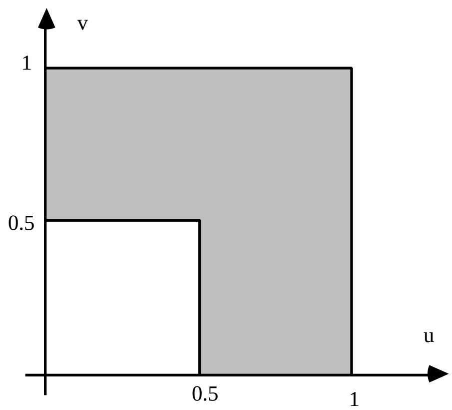
2. The random point ($\mathbf{X}$,$\mathbf{Y}$) is uniformly distributed on the shaded region shown in the left-hand figure below.

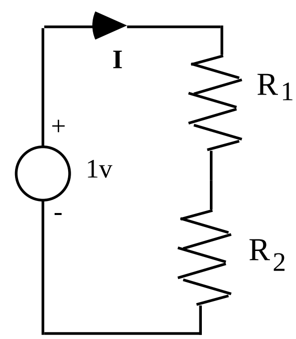
(a) What is the numerical value of $f_{\mathbf{X},\mathbf{Y}}(0.75,0.75)$?

(b) Find the marginal pdf $f_{\mathbf{X}}(u)$ of the random variable $\mathbf{X}$. In order to obtain full credit, you must specify the value of $f_{\mathbf{X}}(u)$ for all real numbers u, $-\infty < u < \infty$.

(c) Find $P\{\mathbf{X} < \mathbf{Y} < 2\mathbf{X}\}$.

(d) Find the pdf of the random variable $\mathbf{Z} = \mathbf{Y}/\mathbf{X}$. In order to obtain full credit, you must specify the value of $f_{\mathbf{Z}}(\alpha)$ for all real numbers α, $-\infty < \alpha < \infty$.





3. Two resistors are connected in series to a one-volt voltage source as shown in the right-hand diagram above. Suppose that the resistance values $\mathbf{R}_1$ and $\mathbf{R}_2$ (measured in ohms) are independent random variables, each uniformly distributed on the interval (0, 1). Find the pdf $f_{\mathbf{I}}(a)$ of the current $\mathbf{I}$ (measured in amperes) in the circuit. Be sure to specify the value of $f_{\mathbf{I}}(a)$ for all real numbers a, $-\infty < a < \infty$.

4. The number of α-particles emitted by a source during a unit time interval can be modeled as a Poisson random variable $\mathbf{X}$ with parameter λ. The α-particles are detected by means of a (imperfect) Geiger counter which detects a particle with probability $p < 1$. The detections of the various particles can be considered to be independent events. Thus, if n particles have been emitted, the Geiger counter reading can be modeled as a binomial random variable $\mathbf{Y}$ with parameters (n,p). In short, $p_{\mathbf{Y}|\mathbf{X}}(k|n)$, the *conditional* pmf of $\mathbf{Y}$ given that $\mathbf{X} = n$, is a binomial pmf: $p_{\mathbf{Y}|\mathbf{X}}(k|n) = \binom{n}{k} p^k(1-p)^{n-k}$ for $0 \le k \le n$. (Note that $\mathbf{Y} \le \mathbf{X}$ always: the counter does not mistakenly count a particle when none is present, i.e. there are no false alarms! but the counter fails to detect a particle with probability p).

(a) Sketch the u-v plane and the joint pmf of $\mathbf{X}$ and $\mathbf{Y}$. Precision is not required in the sizes of the blobs you draw, but be sure that you don't put masses where they do not belong.

(b) What is the unconditional pmf of $\mathbf{Y}$?

(c) What is the conditional pmf of $\mathbf{X}$ given $\mathbf{Y} = k$?

5. **B** and **C** are random variables with joint pdf $f_{\mathbf{B},\mathbf{C}}(u,v) = \begin{cases} u+v, & 0 < u < 1, 0 < v < 1, \\ 0, & \text{elsewhere.} \end{cases}$

(a) Are **B** and **C** are independent random variables?

(b) What is the probability that the time-invariant linear system with transfer function $H(s) = \dfrac{1}{s^2 + 2\mathbf{B}s + \mathbf{C}}$ has an oscillatory impulse response, i.e. an impulse response that takes on both positive and negative values?

6. If **X** is $N(0,\sigma^2)$, then $\mathbf{X}^2$ has gamma pdf with parameter $(1/2, 1/2\sigma^2)$. Now, suppose that **X**, **Y**, and **Z** are independent $N(0,\sigma^2)$ random variables. Then $\mathbf{X}^2$, $\mathbf{Y}^2$, and $\mathbf{Z}^2$ are independent gamma random variables with parameter $(1/2, 1/2\sigma^2)$.

(a) Use the comment immediately following the proof of Proposition 3.1 (pp. 266-267) of Ross to *state* what the *type* of pdf of $\mathbf{W} = \mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2$ is, and write down *explicitly* the exact pdf. What *is* the numerical value of $f_{\mathbf{W}}(5)$ if $\sigma^2 = 4$?

(b) Use LOTUS to prove that $E[\mathbf{W}] = 3\sigma^2$. If you actually evaluated an integral to get this answer, shame on you!

(c) In a physical application, **X**, **Y**, and **Z** represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m. Thus, $\mathbf{H} = (1/2)m\mathbf{W}$ is the kinetic energy of the particle, and an important axiom of statistical mechanics asserts that the average kinetic energy is $E[\mathbf{H}] = E[(1/2)m\mathbf{W}] = (1/2)mE[\mathbf{W}] = (3/2)m\sigma^2 = (3/2)kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in °K. (Note that the average energy is (1/2)kT per dimension.) Show that the kinetic energy **H** has the Maxwell-Boltzmann pdf $f_{\mathbf{H}}(\beta) = \dfrac{2}{\sqrt{\pi}}(kT)^{-3/2}\sqrt{\beta}\exp(-\beta/kT),\ \beta > 0.$

(d) $\mathbf{V} = \sqrt{\mathbf{W}} = \sqrt{\mathbf{X}^2 + \mathbf{Y}^2 + \mathbf{Z}^2}$ is the "speed" of the molecule. Show that the pdf of **V** is

$f_{\mathbf{V}}(\gamma) = \dfrac{4}{\sqrt{\pi}}\left(\dfrac{m}{2kT}\right)^{3/2}\gamma^2\exp\left(-\dfrac{m\gamma^2}{2kT}\right)\ \gamma > 0$ cf. Theoretical Exercise 1, p. 237 of the text.

(e) What is $E[\mathbf{V}]$, the average speed of the molecule?

7. **X** and **Y** are independent random variables uniformly distributed on [0,1].

(a) Find the pdf of $\mathbf{Z} = \mathbf{X} + \mathbf{Y}$.

(b) Let $\mathbf{A} = \min(\mathbf{X},\mathbf{Y})$ and $\mathbf{B} = \max(\mathbf{X},\mathbf{Y})$. Use the results on p. 146 of the Lecture Notes to write down the joint pdf $f_{\mathbf{A},\mathbf{B}}(\alpha,\beta)$. You *should* get a pdf that we have studied in class except, of course, that α and β are being used in place of u and v.

(c) In class, we also found the pdf of **A** + **B** when the joint pdf is as in part (b). Why is the pdf of **A** + **B** so remarkably similar to the pdf of **Z**?