Assigned: Wednesday, October 27, 1999
Due: Wednesday, November 3, 1999
Reading: Ross, Chapters 5 and 6
Noncredit Exercises: Ross, pp. 232-237: 30, 31, 35-39; pp. 237-241: 14, 16, 28, 29; pp. 241-243: 1, 4, 8-10, 13-16.

Problems: Note that LOTUS means Proposition 2.1 on page 197 of Ross

1. **X** is a continuous random variable uniformly distributed on [–1, +1].

(a) If $\mathbf{Y} = \mathbf{X}^2$, what are the mean and variance of **Y**?

(b) If $\mathbf{Z} = g(\mathbf{X})$ where $g(u) = \begin{cases} u^2, & u \geq 0, \\ -u^2, & u < 0, \end{cases}$ use LOTUS to find E[**Z**].

2. A signal **X** is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value **Y** (where $\mathbf{Y} = \alpha$ if $\mathbf{X} > 0$ and $\mathbf{Y} = -\alpha$ if $\mathbf{X} \leq 0$) is used. Note that **Y** is a *discrete* random variable.

(a) What is the pmf of **Y**?

(b) Suppose that $\alpha = 1$. If the signal **X** happens to have value 1.29, what is the error made in representing **X** by **Y**? What is the squared-error? Repeat for the case when **X** happens to have value $\pi/4$ and when **X** happens to have value $-\pi/4$.

(c) We wish to design the quantizer so as to minimize the squared-error. However, since **X** (and **Y**) are random, we can only minimize the squared-error in the probabilistic (that is, average) sense. Now, part (b) shows that the squared-error depends on the value of **X**, and can be expressed as $\mathbf{Z} = (\mathbf{X} - \mathbf{Y})^2 = g(\mathbf{X}) = \begin{cases} (\mathbf{X} - \alpha)^2 & \text{if } \mathbf{X} > 0 \\ (\mathbf{X} + \alpha)^2 & \text{if } \mathbf{X} \leq 0. \end{cases}$

So we want to choose α so that E[**Z**] is as small as possible. Use LOTUS to easily find E[**Z**] as a function of α, and then find the value of α that minimizes E[**Z**].

(d) We now get more ambitious and use a 3-bit A/D converter which first quantizes **X** to the nearest integer **W** in the range –3 to +3. Thus, $\mathbf{W} = 3$ if $\mathbf{X} \geq 2.5$, $\mathbf{W} = 2$ if $1.5 \leq \mathbf{X} < 2.5$, etc. Note that **W** is a discrete random variable. Find the pmf of **W**.

(e) The output of the A/D converter is a 3-bit 2's complement representation of **W**. Suppose that the output is $(\mathbf{Z}_2, \mathbf{Z}_1, \mathbf{Z}_0)$. What is the pmf of $\mathbf{Z}_2$? of $\mathbf{Z}_1$? of $\mathbf{Z}_0$?

(f) Noncredit exercise (but a real-life engineering problem!): Suppose that **W** takes on values $-3\alpha, -2\alpha, -\alpha, 0, +\alpha, +2\alpha, +3\alpha$ and quantization is as before: **X** is mapped to the nearest **W** value. What value of α minimizes $E[(\mathbf{X} - \mathbf{W})^2]$?

3. **X** is a continuous random variable with pdf $f_{\mathbf{X}}(u) = 0.5 \exp(-|u|), \quad -\infty < u < \infty$.

(a) What is the value of $P\{\mathbf{X} \leq \ln 2\}$?

(b) Find the conditional probability that $P\{|\mathbf{X}| \leq \ln 2\}$ given that $\{\mathbf{X} \leq \ln 2\}$.

(c) Now suppose that **X** denotes the voltage applied to a semiconductor diode, and that the current **Y** is given by $\mathbf{Y} = e^{\mathbf{X}} - 1$. Find the pdf of **Y**.

4. **X** is a geometric random variable with parameter 1/2, and $\mathbf{Y} = \sin(\pi\mathbf{X}/2)$.
Is **Y** a continuous random variable or a discrete random variable or a mixed random variable?
If you think that **Y** is a continuous random variable, find the pdf of **Y**.
If you think that **Y** is a discrete random variable, find the pmf of **Y**.
If you think that **Y** is a mixed random variable, find the CDF of **Y**.

5. The random variable **X** has probability density function $f_{\mathbf{X}}(u) = \begin{cases} 2(1-u), & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$

Let $\mathbf{Y} = (1 - \mathbf{X})^2$.

(a) What is the minimum value of **Y**? Call this α.
What is the maximum value of **Y**? Call this β.
What do you think are the values of $P\{\mathbf{Y} \le \alpha - 1\}$ and $P\{\mathbf{Y} > 2\beta\}$?

(b) What is the CDF $F_{\mathbf{Y}}(v)$ of the random variable **Y**?
Be sure to specify the value of $F_{\mathbf{Y}}(v)$ for all v, $-\infty < v < \infty$.

(c) Show that the CDF $F_{\mathbf{Y}}(v)$ that you found in part (b) is a nondecreasing continuous function. Is $F_{\mathbf{Y}}(v)$ differentiable at α? at β?

(d) From the definition of the CDF $F_{\mathbf{Y}}(v)$, we know that $P\{\mathbf{Y} \le \alpha - 1\} = F_{\mathbf{Y}}(\alpha - 1)$ and $P\{\mathbf{Y} > 2\beta\} = 1 - F_{\mathbf{Y}}(2\beta)$. Does substituting $v = \alpha - 1$ and $v = 2\beta$ in the CDF $F_{\mathbf{Y}}(v)$ that you found in part (b) give the same values for $P\{\mathbf{Y} \le \alpha - 1\}$ and $P\{\mathbf{Y} > 2\beta\}$ that you stated in part (a)? If the values are different, which ones are the correct values? Explain your choices of correct answer (and why the other possible answers are wrong) in detail.

6. The radius of a sphere is a random variable **R** with pdf $f_{\mathbf{R}}(\rho) = \begin{cases} 3\rho^2, & 0 < \rho < 1, \\ 0 & \text{elsewhere.} \end{cases}$

(a) Find the CDF $F_{\mathbf{V}}(\alpha)$ and pdf $f_{\mathbf{V}}(\alpha)$ of **V**, the volume of the sphere.

(b) If the sphere is made of metal and carries an electrical charge of Q coulombs, what is the CDF $F_{\mathbf{S}}(x)$ and the pdf $f_{\mathbf{S}}(x)$ of the surface charge density **S** on the sphere ?

7. The lifetime of a system with hazard rate $\lambda(t) = bt$ is a Rayleigh random variable **X** with pdf $f(u) = (bu)\bullet\exp(-bu^2/2)$ for $u > 0$ (Ross, p. 221). The complementary CDF is given by $P\{\mathbf{X} > t\} = \exp(-bt^2/2)$ for $t > 0$.

(a) Find the mean lifetime $E[\mathbf{X}]$ of the system using the formula $E[\mathbf{X}] = \int_0^\infty u\bullet f(u)du$.

(b) Use the result $E[\mathbf{X}] = \int_0^\infty P\{\mathbf{X} > t\}dt$ to find the mean lifetime of the system. Do you get the same answer as in part (a)? Why or why not?

(c) What is the median lifetime, and is it larger or smaller than the mean lifetime? How do these parameters compare to the mode of lifetime (mode = location of the pdf maximum)

(d) The system fails at time t, i.e. $\mathbf{X} = t$ is observed to have occurred on this trial. What is the maximum-likelihood estimate of the parameter b that occurs in the pdf and hazard rate? Remember that the maximum-likelihood estimate $\hat{b}$ of the parameter b maximizes the pdf at the observed value t. Thus, for given t, what value of b maximizes $(bt)\exp(-bt^2/2)$?