Assigned: Wednesday, September 29, 1999
Due: Wednesday, October 6, 1999
Reading: Ross, Chapter 4.1, 4.3–4.5, 4.7- 4.8, 4.9.1-4.9.3 and Chapter 3
Noncredit Exercises: (Do not turn these in) Ross pp. 104-117: 1, 2, 5, 10, 12, 16, 31, 38, 39, 44; pp. 118-122: 4, 23; p. 123: 2,3; pp.173-184: 4, 12, 14, 21, 24, 33, 35, 38, 39, 47, 57, 60, 69; pp. 184-188: 4.
Problems:

1. [The Once and Future King] You are trying to persuade an empty-headed monarch that you can foresee the future. You offer to forecast what happens on repeated independent tosses of a biased coin of the realm which you happen to know has P(Heads) = 0.11.

(a) The skeptical king asks you to predict the number of heads that will occur on the next 1000 tosses and promises to execute you if your guess is wrong, just to make it more interesting. Which number should you predict and why? What is the probability that the 1000 coin tosses *do* result in the number of heads you predicted?

(b) You luck out and guess right in part (a). The next day, the king asks you to predict how many tosses will be required to observe the next Head. Which number should you predict and why? What is the probability that a Head *does* occur for the frst time on the toss you predicted?

(c) Since you guessed right twice in a row, the king is thinking that you can indeed see into the future, and assigns a harder problem: predict the number of tosses required to observe a Head for the 105th time. Which number should you predict and why? What is the probability that a Head *does* occur for the 105th time on the toss you predicted?

Courtiers jealous of your growing fame substitute a coin bearing an image of the king's father. The new coin has a different probability of Heads. Fortunately, this is observed by your trusty sidekick who tells you that the coin to be used tomorrow is different. Naturally you are reluctant to make further predictions about the coin. To forestall further requests for amazing demonstrations of your powers, you tell the king that you have the power to estimate probabilities from experimental data, and the king, who flunked out of ECE 313, is duly impressed. He tells you that he is going to toss the coin 1000 times and that you are to estimate P(Heads) = p.

(d) Heads occurred for the frst time on the 12th toss. You consider announcing the value of p right away (without waiting for the 1000 tosses to be completed). What is the maximum-likelihood estimate of the value of p? that is, what value of p maximizes the probability of a Head occurring for the first time on the 12th toss?

(e) You decide that maybe it is best to wait for the results of some more tosses before deciding on your estimate of p. The 300th head occurred on the 994th toss. What value of p maximizes the probability of a Head occurring for the 300th time on the 994th toss?

(f) You sensibly decide to wait out the last 6 tosses also, and all of these result in Tails. What is your estimate of the value of p after 1000 tosses?

(g) You already knew after 994 tosses that 300 heads occurred. The last 6 tosses did not result in Heads and thus conveyed no information about p. So why isn't the maximum-likelihood estimate of p in part (f) the same as the estimate in part (e)?

2. You believe that the Fighting Illini (or the Bears or the Blackhawks or the Cubs if you prefer) have probability p of winning a game, where each game can be considered to be an independent trial. Being a fair-minded pessimist, you want to place a fair bet (that is, one that you have a roughly 50% chance of winning) on the length of the losing streak. You bet on a number, and you win if the losing streak is *at least* that long. What number should you bet on? (It is a function of p, of course!). The number you will find is the *median* length of losing streaks. Roughly 1/2 of the losing streaks will be longer and roughly half will be shorter than this number.

3. Consider a Poisson process consisting of balls coming out of a hole. The rate for this process is λ. Coming out of the hole, the balls are sent either down tube 2 or tube 3, according to the following criterion. For each ball, a fair coin is flipped n times. If there are exactly m heads in the first n flips, (where $m < n$), then the ball is sent down tube 2. Otherwise, it is sent down tube 3.

(a) Find the expected value for the number of balls arriving into tube 3 in an interval of length t.

(b) Suppose that n is 10. Determine m to maximize your answer to part (a).

4. **Ross,** exercises 6 on page 185.

5. I toss a fair coin 100 times. Consider the events A = I get 10 Heads in the first 20 tosses, B = I get at least one Tails in the first 10 tosses, C = I get 3 Heads in the last 50 tosses, D = I get at least 5 Tails in tosses 25-30. Write the probability of each event. Write all possible sets of mutually independent events.