1.(a) $E[\mathbf{Y}] = E[\mathbf{X}^2] = \int_{-1}^{+1} u^2(1/2)du = 1/3$. $E[\mathbf{Y}^2] = E[\mathbf{X}^4] = \int_{-1}^{+1} u^4(1/2)du = 1/5$.

Hence, $\text{var}(\mathbf{Y}) = E[\mathbf{Y}^2] - (E[\mathbf{Y}])^2 = (1/5) - (1/3)^2 = 4/45$.

(b) $E[\mathbf{Z}] = \int_{-\infty}^{\infty} g(u)f(u)du = \int_{-1}^{0} -u^2(1/2)du + \int_{0}^{1} u^2(1/2)du = -1/3 + 1/3 = 0$.

2.(a) Obviously $P\{\mathbf{Y} = \alpha\} = P\{\mathbf{Y} = -\alpha\} = 1/2$.

(b) $(1.29-1) = 0.29$. $(1.29-1)^2 = 0.0841$. $(\pi/4-1) = -0.214\ldots$, $(\pi/4-1)^2 = 0.046\ldots$.
$(-\pi/4-(-1)) = -0.214\ldots$, $(-\pi/4-(-1))^2 = 0.046\ldots$. Note that the error for $+\mathbf{X}$ is the same as that for $-\mathbf{X}$.

(c) $E[\mathbf{Z}] = \int_{0}^{\infty}(u-\alpha)^2 f(u)du + \int_{-\infty}^{0}(u+\alpha)^2 f(u)du = \int_{-\infty}^{\infty}(u^2 + \alpha^2)f(u)du - 4\alpha\int_{0}^{\infty} uf(u)du = 1+\alpha^2-4\alpha/\sqrt{2\pi}$ on expanding out the quadratics, changing variables, and using the fact that $E[\mathbf{X}^2] = \sigma^2 + \mu^2 = 1$. Note that $uf(u)$ is a perfect integral. It is easy to show that $E[\mathbf{Z}]$ has minimum value $1-2/\pi$ at $\alpha = \sqrt{2/\pi}$

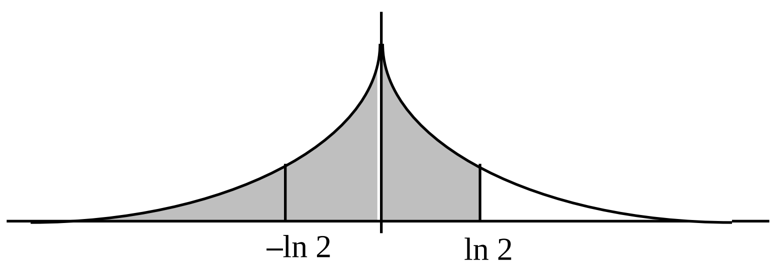
(d) From tables of $\Phi(\bullet)$, we get $P\{\mathbf{W} = -3\} = P\{\mathbf{W} = +3\} = \Phi(-2.5) = 0.0062$,
$P\{\mathbf{W} = 0\} = \Phi(0.5) - \Phi(-0.5) = 0.3830$, $P\{\mathbf{W} = -1\} = P\{\mathbf{W} = +1\} = \Phi(1.5) - \Phi(0.5) = 0.2417$, and
$P\{\mathbf{W} = -2\} = P\{\mathbf{W} = +2\} = \Phi(2.5) - \Phi(1.5) = 0.0606$.

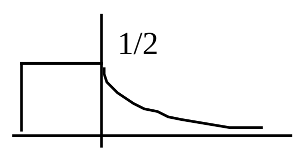
(e) $P\{\mathbf{Z}_2 = 1\} = P\{\mathbf{W} < 0\} = 0.3085$.
$P\{\mathbf{Z}_1 = 1\} = P\{\mathbf{W} = 2\} + P\{\mathbf{W} = 3\} + P\{\mathbf{W} = -1\} + P\{\mathbf{W} = -2\} = 0.3691$
$P\{\mathbf{Z}_0 = 1\} = P\{\mathbf{W} = 2\} + P\{\mathbf{W} = 0\} + P\{\mathbf{W} = -2\} = 0.5042 \neq P\{\mathbf{Z}_0 = 0\}$

3.(a) From the figure shown below, we see that the pdf is symmetric about u = 0. Hence, we get that

$$P\{\mathbf{X} \le \ln 2\} = \frac{1}{2} + \int_0^{\ln 2} \frac{1}{2}\exp(-u)\,du = \frac{1}{2} + \left[-\frac{1}{2}\exp(-u)\right]_0^{\ln 2} = \frac{3}{4}.$$ Notice that $P\{0 \le \mathbf{X} \le \ln 2\} = \frac{1}{4}$

(b) $$P\{|\mathbf{X}| \le \ln 2 \mid \mathbf{X} \le \ln 2\} = \frac{P\{\{|\mathbf{X}| \le \ln 2\} \cap \{\mathbf{X} \le \ln 2\}\}}{P\{\mathbf{X} \le \ln 2\}} = \frac{P\{|\mathbf{X}| \le \ln 2\}}{P\{\mathbf{X} \le \ln 2\}} = \frac{2P\{0 \le \mathbf{X} \le \ln 2\}}{3/4} = \frac{1/2}{3/4} = \frac{2}{3}.$$





(c) The minimum value of $\mathbf{I}$ is -1. $F_{\mathbf{I}}(b) = P\{\mathbf{I} \le b\} = 0$ if $b < -1$.
For $b \ge -1$, $F_{\mathbf{I}}(b) = P\{\mathbf{I} \le b\} = P\{e^{\mathbf{V}} - 1 \le b\} = P\{\mathbf{V} \le \ln(b+1)\} = F_{\mathbf{V}}(\ln(b+1))$. But,

$$F_{\mathbf{V}}(a) = \begin{cases} e^{a}/2, & a \le 0, \\ 1 - e^{-a}/2, & a > 0. \end{cases} \quad \text{Thus, } F_{\mathbf{I}}(b) = \begin{cases} \frac{b+1}{2} & -1 \le b \le 0, \\ 1 - \frac{1}{2(b+1)}, & b > 0, \end{cases}$$

and $$f_{\mathbf{I}}(b) = \begin{cases} \frac{1}{2} & -1 \le b \le 0, \\ \frac{1}{2(b+1)^2}, & b > 0. \end{cases}$$ Note that the pdf has constant value 1/2 in the range $-1 \le b \le 0$.

4. Since $\mathbf{X}$ is discrete, so is $\mathbf{Y}$. Since $\mathbf{X}$ takes on integer values, we have that

$$\mathbf{Y} = \sin(\pi\mathbf{X}/2) = \begin{cases} 0, & \text{if } \mathbf{X} \text{ is even,} \\ +1, & \text{if } \mathbf{X} = 4k+1, \quad k = 0, 1, \ldots \\ -1, & \text{if } \mathbf{X} = 4k+3, \quad k = 0, 1, \ldots \end{cases}$$

and hence, $p_{\mathbf{Y}}(0) = P\{\mathbf{X} \text{ is even}\} = \sum_{k=1}^{\infty} P\{\mathbf{X} = 2k\} = \sum_{k=1}^{\infty} (1/2)^{2k} = (1/4)\sum_{k=0}^{\infty}(1/4)^k = 1/3$

$p_{\mathbf{Y}}(1) = \sum_{k=0}^{\infty} P\{\mathbf{X} = 4k+1\} = \sum_{k=0}^{\infty} (1/2)^{4k+1} = (1/2)\sum_{k=0}^{\infty}(1/16)^k = 8/15$

$p_{\mathbf{Y}}(-1) = \sum_{k=0}^{\infty} P\{\mathbf{X} = 4k+3\} = \sum_{k=0}^{\infty} (1/2)^{4k+3} = (1/8)\sum_{k=0}^{\infty}(1/16)^k = 2/15$

5.(a) Since $\mathbf{X}$ takes on values between 0 and 1, so does $\mathbf{Y}$ and thus $\alpha = 0$ and $\beta = 1$. Obviously, $P\{\mathbf{Y} \le \alpha-1\} = P\{\mathbf{Y} \le -1\} = 0$ and $P\{\mathbf{Y} > 2\beta\} = P\{\mathbf{Y} > 2\} = 0$.

(b) Let $0 \le v \le 1$. Then, $F_{\mathbf{Y}}(v) = P\{\mathbf{Y} \le v\} = P\{(1-\mathbf{X})^2 \le v\} = P\{-\sqrt{v} \le 1 - \mathbf{X} \le \sqrt{v}\} = P\{\mathbf{X} \ge 1 - \sqrt{v}\}$ $= 1 - F_{\mathbf{X}}(1-\sqrt{v}) = (1 - (1-\sqrt{v}))^2 = v$ where we used the result that $F_{\mathbf{X}}(u) = 1 - (1-u)^2$.

Hence, $F_{\mathbf{Y}}(v) = \begin{cases} 0, & u < 0, \\ v, & 0 \le v \le 1, \\ 1, & v > 1. \end{cases}$

(c) A sketch of the function $F_{\mathbf{Y}}(v)$ reveals that it is a nondecreasing continuous function. It is not differentiable at $\alpha = 0$ or at $\beta = 1$.

(d) Yes, we get the same values for $P\{\mathbf{Y} \le \alpha-1\}$ and $P\{\mathbf{Y} > 2\beta\}$ as in part (a).

6.(a) The volume V has values in the range $(0, 4\pi/3)$. For any u, $0 < u < 4\pi/3$, $F_{\mathbf{V}}(u) = P\{\mathbf{V} \le u\}$ $= P\{4\pi\mathbf{R}^3/3 \le u\} = P\{\mathbf{R} \le \sqrt[3]{3u/4\pi}\} = F_{\mathbf{R}}(\sqrt[3]{3u/4\pi}) = 3u/4\pi$ since $F_{\mathbf{R}}(\rho) = \rho^3$ for $0 < \rho < 1$. Hence, $f_{\mathbf{V}}(u)$ is uniform on $(0, 4\pi/3)$.

(b) The electrical charge is uniformly distributed on the surface of the sphere. The surface charge density is $\mathbf{S} = Q/4\pi\mathbf{R}^2 > Q/4\pi$.

For $x > Q/4\pi$, $F_{\mathbf{S}}(x) = P\{\mathbf{S} \le x\} = P\{Q/4\pi\mathbf{R}^2 \le x\} = P\{1 > \mathbf{R} \ge \sqrt{Q/4\pi x}\} = 1 - (Q/4\pi x)^{1.5}$.

Hence, $f_{\mathbf{S}}(x) = (3/2x)(Q/4\pi x)^{1.5}$ for $x > Q/4\pi$, and 0 otherwise.

7.(a) $E[\mathbf{X}] = \int_0^{\infty} u\, f(u)\, du = \int_0^{\infty} u\,(bu)\bullet\exp(-bu^2/2)du = \sqrt{2/b}\int_0^{\infty}\sqrt{v}\bullet\exp(-v)dv$ on setting $bu^2/2 = v$. The integral is $\Gamma(3/2) = (1/2)\Gamma(1/2) = \sqrt{\pi/4}$ and hence $E[\mathbf{X}] = \sqrt{2/b}\,\sqrt{\pi/4} = \sqrt{\pi/2}\sqrt{1/b}$.

(b) $E[\mathbf{X}] = \int_0^{\infty} P\{\mathbf{X} > t\}\, dt = \int_0^{\infty} \exp(-bt^2/2)\, dt$. But, we know (we do? how on earth did I ever get that idea?) that $\int_0^{\infty} (\sqrt{b/2\pi})\bullet\exp(-bt^2/2)\, dt = 1/2$ and hence $E[\mathbf{X}] = \sqrt{\pi/2}\sqrt{1/b}$ just as in part (a).

(c) The median lifetime is T satisfying $P\{\mathbf{X} > T\} = \exp(-bT^2/2) = 1/2$. This gives $T = \sqrt{2\ln 2}\sqrt{1/b}$. Since $2 \ln 2 < \pi/2$, the median is smaller than the mean. The mode of the pdf is easily found to be at $\sqrt{1/b}$ and is the smallest of the three central measures.

(d) $\frac{d}{db}(bt)\exp(-bt^2/2) = t\bullet\exp(-bt^2/2) - bt\bullet\exp(-bt^2/2)\bullet t^2/2$ is zero for $b = \sqrt{2}/t$. Thus, if we observe that $\mathbf{X} = t$, the maximum-likelihood estimate of b is $\sqrt{2}/t$. Reality check: If t is large, we estimate the value of b to be quite small. This makes sense. If the system lasted for a long time, its hazard rate can be expected to be small, and the hazard rate is proportional to b.