1.(a) Yes. **(b)** We interpret both -2^2+1 and $1-2^2$ as having value -3, and -2^3+2 and $2-2^3$ as having value -6. However, American grade–school math teachers (and their students!) may have a different interpretation! According to the Microsoft Excel manual, the – sign in -2^2+1 denotes *unary* negation of 2 which has higher priority than the squaring(exponentiation) of 2. Thus, -2^2+1 means $(-2)^2+1 = 5$; while the – sign in $1-2^2$ denotes the binary *subtraction* operator which has a lower priority than exponentiation, so that $1-2^2 = -3$. Note that with this interpretation, both -2^3+2 and $2-2^3$ have value -6.

Comment: Microsoft's world view invalidates many common mathematical operations. For example, you cannot factor as in $-2^3+2 = (-2^2+1)2$ and $2-2^3 = (1-2^2)2$ because the cancellation law ($ac = bc$ implies $a = b$ except when $c = 0$) would then imply that $-2^2+1 = 1-2^2$ instead of having different values 5 and -3 respectively!

Moral: (To the tune of Jingle Bells)

Parentheses, exponents, always do them first.
Multiply and divide, this you must rehearse.
Then you add and subtract, that's the very last.
Do all these from left to right and you will surely pass.

Later on, when teaching you about Gaussian random variables, we will be writing $\exp(-x^2)$ a lot, so be sure to remember that we mean $\exp(-(x^2))$ and not $\exp((-x)^2) = \exp(x^2)$! Also, be very careful if and when you use Microsoft Excel in statistical applications!

(d) True: $\dfrac{1}{-\alpha \pm \sqrt{\beta}} = \dfrac{1}{-\alpha \pm \sqrt{\beta}} \bullet \dfrac{-\alpha \mp \sqrt{\beta}}{-\alpha \mp \sqrt{\beta}} = \dfrac{-\alpha \mp \sqrt{\beta}}{\alpha^2 - \beta^2}$ showing that $\dfrac{1}{-b \pm \sqrt{b^2 - 4ac}} = \dfrac{-b \mp \sqrt{b^2 - 4ac}}{4ac}$

It follows that the given expression simplifies to the more familiar $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

2.(b) $\cot(15)\cot(35)\cot(55)\cot(75) = 1$. For any x, $0 \le x < 45$, $\cot(45-x)\cot(45+x) = 1$, and hence if x and y are integers such that $0 < x < y < 45$, we get that $0 < 45-y < 45-x < 45+x < 45+y < 90$, and $\cot(45-y)\cot(45-x)\cot(45+x)\cot(45+y) = 1$. It is not quite as easy to find such formulas if the RHS is required to be an integer other than 1 (as in the surprising result of part (a)). A proof of the result of part (a) is as follows. $\cos(a-b) + \cos(a+b) = 2 \cos a \cos b$, while $\cos (a-b) - \cos (a+b) = 2 \sin a \sin b$.
Also, $\cos 4x = 8 \cos^4 x - 8 \cos^2 x + 1$, and $\cos 3x = 4 \cos^3 x - 3 \cos x$.

Hence, $\cot(10)\cot(30)\cot(50)\cot(70) = \dfrac{(\cos 80 + \cos 60)(\cos 80 + \cos 20)}{(\cos 80 - \cos 60)(\cos 80 - \cos 20)}$.

Let $\cos 20 = y$. Since $\cos 60 = 1/2$, we know that $4y^3-3y = 1/2$, and that $\cos 80 = 8y^4 - 8y^2 + 1$ $= 2y(4y^3 - 3y) - 2y^2 + 1 = 1 + y - 2y^2$. Thus, the above fraction equals

$$\frac{(3/2 + y - 2y^2)(1 + 2y - 2y^2)}{(1/2 + y - 2y^2)(1 - 2y^2)} = \frac{3/2 + 4y - 3y^2 - 6y^3 + 4y^4}{1/2 + y - 3y^2 - 2y^3 + 4y^4} = \frac{3/4}{1/4} = 3$$

on substituting $y^3 = (3/4)y + 1/8$ and $4y^4 = 3y^2 + y/2$.

(c) Since $\dfrac{1}{\sqrt{n+1} + \sqrt{n}} = \dfrac{1}{\sqrt{n+1} + \sqrt{n}} \bullet \dfrac{\sqrt{n+1} - \sqrt{n}}{\sqrt{n+1} - \sqrt{n}} = \dfrac{\sqrt{n+1} - \sqrt{n}}{(n+1) - (n)} = \sqrt{n+1} - \sqrt{n}$, the sum "telescopes" to $\sqrt{100} - \sqrt{1} = 9$. Look, Ma! No calculator.

3.(a) 7. **(b)** My cheap calculator gives 0 as the answer in all cases. This seems reasonable because $\sin x \approx x$ for small x, and thus the function should have value nearly 0 as x approaches 0. However, the answers that you get *will* depend on the number of digits of precision used by your calculating device as well as the algorithm it uses to compute $\sin(x)$. For example, at $x = 10^{-7}$, the Unix high-precision calculator utility *bc* gives the following values: $\approx 10^9$ using 20 digits; $\approx 10^3$ using 25 digits, ≈ 0.34 using 30 digits and ≈ 0.33333333333334 using 45 or more digits. Thus, it would appear that the limit might be 1/3. For the formal proof, note $\sin x = x - x^3/3! + x^5/5! - \ldots \approx x - x^3/6$ for small x. Hence, for small x,

$$\frac{1}{[\sin x]^2} - \frac{1}{x^2} \approx \frac{1}{x^2}\left[\frac{1}{(1 - x^2/6)^2} - 1\right] = \frac{1}{x^2}\left[1 + 2x^2/6 + 3(x^2/6)^2 + \ldots - 1\right] = \frac{1}{3} + \frac{3x^2}{36} \ldots$$

Here we have used the series: $(1 - y)^{-2} = 1 + 2y + 3y^2 + \ldots$ which is valid for $|y| < 1$.
Remember this series; it and the series $(1 - y)^{-1} = 1 + y + y^2 + y^3 + \ldots$ will be used many times in the course.
Exercise: What do you get when you differentiate both sides of $(1 - y)^{-1} = 1 + y + y^2 + y^3 + \ldots$

Thus, $\frac{1}{[\sin x]^2} - \frac{1}{x^2} \approx 1/3 + 3x^2/36 + \ldots = 1/3$ at x = 0, i.e., $\lim_{x\to 0} \frac{1}{[\sin x]^2} - \frac{1}{x^2} = \frac{1}{3}$

Those who prefer a purer approach should note that $\frac{1}{[\sin x]^2} - \frac{1}{x^2} = \frac{x^2 - \sin^2 x}{x^2 \sin^2 x}$ is of the form 0/0 at x = 0.

Thus, by (repeated application of) L'Hôpital's rule and the identity $\frac{d}{dx}\sin^2 x = \sin(2x)$ (huh??), we have

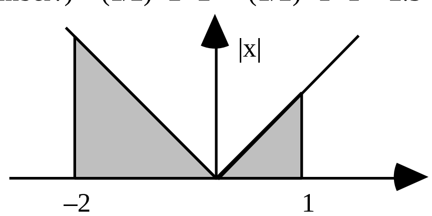
$$\lim_{x\to 0} \frac{1}{[\sin x]^2} - \frac{1}{x^2} = \lim_{x\to 0} \frac{x^2 - \sin^2 x}{x^2 \sin^2 x} = \lim_{x\to 0} \frac{2x - \sin(2x)}{2x\sin^2 x + x^2\sin(2x)}$$

$$= \lim_{x\to 0} \frac{2 - 2\cos(2x)}{2\sin^2 x + 4x\sin(2x) + 2x^2\cos(2x)} = \lim_{x\to 0} \frac{4\sin(2x)}{2\sin(2x) + 12x\cos(2x) - 4x^2\sin(2x) + 4\sin(2x)}$$

$$= \lim_{x\to 0} \frac{8\cos(2x)}{4\cos(2x) + 12\cos(2x) + 8\cos(2x) + x(\text{some other stuff})} = \frac{8}{24} = \frac{1}{3}$$ just as before.

(c) $x^n\exp(-ax)$ has maximum value $(n/a)^n\exp(-n)$ at x = n/a. Here n = 25, a = ln 1.0001 giving a maximum value of $1.2350\ldots\times 10^{124}$ at x = 250012.49979...

4.(a) From the picture: $\int_{-2}^{1} |x|dx$ = the shaded area = sum of the areas of two triangles (area = (1/2)×base×height, remember?) = (1/2)×2×2 + (1/2)×1×1 = 2.5



Those who insist on integration should note that |x| = x if x > 0 and |x| = –x if x < 0, and hence $\int_{-2}^{1} |x|dx = \int_{-2}^{0} -x\,dx + \int_{0}^{1} x\,dx = \left.\frac{-x^2}{2}\right|_{-2}^{0} + \left.\frac{x^2}{2}\right|_{0}^{1} = 2.5$. The answer is NOT $\int_{-2}^{1} |x|dx = \left.\frac{|x|^2}{2}\right|_{-2}^{1} = \frac{1^2}{2} - \frac{|-2|^2}{2} = -1.5$

It is very important to learn the habit of drawing a sketch before tackling integrals in this course. It will save you from a lot of errors, and in some cases, it will ease your task considerably. The above is an example of this.

Put y = 1 – x. Then, $\int_{-2}^{1} x(1-x)^{19}dx = \int_{3}^{0} (1-y)y^{19}(-dy) = \int_{0}^{3} y^{19} - y^{20}\,dy = \frac{3^{20}}{20} - \frac{3^{21}}{21} = -\frac{13\times 3^{20}}{140}$ $= -3.237728\ldots\times 10^8$. Alternatively, integration by parts gives the same result.

(b) Since an integral is “the area under the curve” the integral of a nonnegative function cannot be negative. A proof of this can be written using (for example) the comparison property (see, e.g. Edwards and Penney).

I hope that warning bells will go off in your mind if you ever find the integral of a positive function to be a negative number. These bells will peal out more strongly if your sixth sense is aided by a sketch indicating just which area corresponds to the integral that you are computing.

(c) **(i)** False: the chain rule gives –g(–x) as the derivative. **(ii)** True as per the chain rule. **(iii)** False: the chain rule gives $\exp(f(x^2))g(x^2)2x$ as the derivative **(iv)** True. cf. (i) **(v)** False: the antiderivative of $g(x^2)$ need not be related to $f(x^2)$ at all! **(vi)** True only for positive functions f(x).

(d) Since $\frac{d}{dx}\exp(-x^2/2) = -x\cdot\exp(-x^2/2)$, $\int_{1}^{\infty} x\cdot\exp(-x^2/2)dx = \left.-\exp(-x^2/2)\right|_{1}^{\infty} = \exp(-1)$.

5.(a) $\frac{d}{dx}\arctan(x) = \frac{1}{1+x^2}$. **(b)** $I = \int_{-1}^{1} \frac{2}{1+x^2}\,dx = \left.2\cdot\arctan(x)\right|_{-1}^{1} = 2\cdot[\pi/4 - (-\pi/4)] = \pi$.

(c) True. The value of the integral does not depend on the name of the variable of integration.

(d)(e) The substitution $y = 1/x$, $dx = -y^{-2}\,dy$, $x = -1$ $y = -1$, $x = 1$ $y = 1$ only **seemingly** leads to the result that $I = -J$. In fact, although it is perfectly true that $x = -1$ $y = -1$, it so happens that as x **increases** from –1 to 0, y **decreases** from –1 to –∞. Thus, the change of variables really gives

$$I = \int_{-1}^{1}\frac{2}{1+x^2}\,dx = \int_{-1}^{0}\frac{2}{1+x^2}\,dx + \int_{0}^{1}\frac{2}{1+x^2}\,dx = \int_{-1}^{-\infty}\frac{-2}{1+y^2}\,dy + \int_{\infty}^{1}\frac{-2}{1+y^2}\,dy = \int_{-\infty}^{-1}\frac{2}{1+y^2}\,dy + \int_{1}^{\infty}\frac{2}{1+y^2}\,dy$$

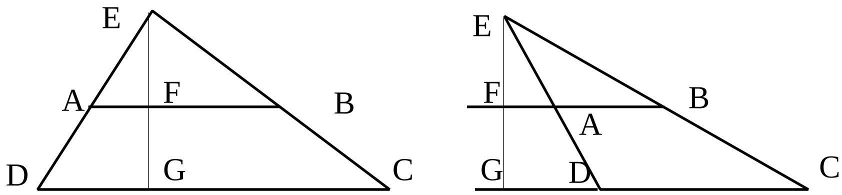
$$= 2\bullet\arctan(x)\Big|_{-\infty}^{-1} + 2\bullet\arctan(x)\Big|_{1}^{\infty} = 2\bullet[(-\pi/4) - (-\pi/2)] + 2[(\pi/2) - (\pi/4)] = \pi \text{ just as before.}$$

Sorry, folks, and don't forget to re-change all those 0's back to π's in your textbooks!

6. From the figures shown below, we get that
Area(ABCD) = Area (EDC) – Area(EAB) = (1/2) × (DC) × (EG) – (1/2) × (AB) × (EF).
Add and subtract (1/2) × (DC) × (EF) – (1/2) × (AB) × (EG) on the right hand side to get

Area(ABCD) = (1/2) × (DC) × (EG) – (1/2) × (AB) × (EF) – (1/2) × (AB) × (EG)
+ (1/2) × (AB) × (EG) – (1/2) × (DC) × (EF) + (1/2) × (DC) × (EF)
= (1/2) × (DC+AB) × (EG) – (1/2)(DC+AB) × (EF)
– (1/2) × (AB) × (EG) + (1/2) × (DC) × (EF)
= (1/2) × (DC+AB) × (FG) – (1/2) × (AB) × (EG) + (1/2) × (DC) × (EF)

But, since EAB and EDC are similar triangles, AB/DC = EF/EG and hence the last two terms sum to 0 showing that Area(ABCD) = (1/2) × (DC+AB) × (FG)
= (1/2) × (sum of the lengths of the parallel sides) × distance between the parallel sides



7.(a)(i) The surface described by f(x,y) is a pyramid of height 1/2 on a square base. If we fix y = 0.25, then f(x,0.25) simplifies to

$$g(x) = f(x,0.25) = \begin{cases} x, & 0 < x \le 1/2,\ x \le 0.25 \le 1-x, \\ 1-x, & 1/2 < x \le 1,\ 1-x \le 0.25 \le x, \\ 0.25, & 0.25 < x < 1-0.25, \\ 0, & \text{elsewhere.} \end{cases} = \begin{cases} x, & 0 < x \le 0.25 \\ 1-x, & 0.75 \le x \le 1, \\ 0.25, & 0.25 < x < 0.75, \\ 0, & \text{elsewhere.} \end{cases}$$

which describes a trapezium of area (1/2) × (1 + 0.5) × 0.25 = 3/16.

(ii) Recall that the volume of a pyramid is (1/3) × base area × height which gives(1/3) × 1 × (1/2) = 1/6 in this case. Otherwise, we have to set up and evaluate 4 different integrals, but a little thought shows that from symmetry considerations, we get $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dy\,dx = 4\int_{x=0}^{1/2}\int_{y=x}^{1-x} x\,dy\,dx = 4\int_{x=0}^{1/2} x(1-2x)\,dx = 1/6$ as before.

(b) Changing to polar coordinates $\int\int (x^2 + y^2)^{-2}\,dxdy = \int_{\sqrt{2}}^{\infty}\int_{0}^{2\pi} r^{-4} r\,d\theta\,dr = 2\pi\int_{\sqrt{2}}^{\infty} r^{-3}\,dr = \pi/2.$