Two different versions of the exam were used: numerical values of the answers depend on the version.

1. There are \( n \) red and \( n \) black balls in the urn.
   (a) \( (R_1 \cup R_2) \cap (B_1 \cup B_2) = (R_1 \cap B_2) \cup (B_1 \cap R_2) \) since \( R_1 \cap B_1 = R_2 \cap B_2 = \emptyset \). Hence,
   \[
P((R_1 \cup R_2) \cap (B_1 \cup B_2)) = P(R_1)P(B_2 | R_1) + P(B_1)P(R_2 | B_1) = \frac{n}{2n} \times \frac{n}{2n-1} + \frac{n}{2n} \times \frac{n}{2n-1} = \frac{n}{2n-1}.
   \]
   \[
P[B_1 | (R_1 \cup R_2)] = \frac{P[B_1 \cap (R_1 \cup R_2)]}{P[R_1 \cup R_2]} = \frac{P[B_1 \cap (R_1 \cup R_2)]}{P[R_1 \cup R_2]}. \quad \text{We have already found}
   \]
   \[
P[R_1 \cup R_2] = \frac{n}{2n} \times \frac{n}{2n-1}. \quad \text{Since \( P[R_1 \cup R_2] = 1 - P[B_1 \cap B_2] = 1 - \frac{n}{2n} \times \frac{n-1}{2n-1} = \frac{n(3n-1)}{2n(2n-1)} \)
   \]
   we get that \( P[B_1 | (R_1 \cup R_2)] = \frac{n}{3n-1} \).
   (b) \( R_1 \) and \( R_2 \) are neither disjoint nor independent. However,
   \[
P(R_2) = P(R_2 | B_1) P(B_1) + P(R_2 | B_2) P(B_2) = \frac{n}{2n} \times \frac{n-1}{2n-1} + \frac{n}{2n} \times \frac{n}{2n-1} = \frac{1}{2} = P(R_1).
   \]

2. Denote by \( \alpha \) the common value of \( P[\text{High} | \text{input} = 1] \) and \( P[\text{Low} | \text{input} = 0] \).
   (a),(b) The likelihood matrix is shown below on the left and the joint probability matrix on the right.

   \[
   \begin{array}{c|cc}
   \text{input} = 0 & \alpha & 1 - \alpha \\
   \hline
   \text{input} = 1 & 1 - \alpha & \alpha
   \end{array}
   \quad \begin{array}{c|c|c}
   \Pi_0 \alpha & \Pi_0 (1 - \alpha) & \Pi_1 \alpha \\
   \hline
   \Pi_0 \alpha & \Pi_1 (1 - \alpha) & \Pi_1 \alpha
   \end{array}
   \]

   Since \( \alpha > 1 - \alpha \), the ML decision rule is as indicated by the shaded squares in the likelihood matrix, i.e.,
   the ML decision rule is \( \left\{ \begin{array}{ll}
   \text{If output = High, decide that a 1 was being transmitted.} \\
   \text{If output = Low, decide that a 0 was being transmitted.}
   \end{array} \right. \)
   If input = 0, the ML decision is in error if output = High, which occurs with probability \( 1 - \alpha \).
   If input = 1, the ML decision is in error if output = Low, which occurs with probability \( 1 - \alpha \).
   Hence, \( P_{e,\text{ML}} = (1 - \alpha)\Pi_0 + (1 - \alpha)\Pi_1 = (1 - \alpha) \).
   (b) \( \Pi_0 \alpha > \Pi_1 (1 - \alpha) \) and \( \Pi_0 (1 - \alpha) > \Pi_1 \alpha \) for the given data and hence the MAP rule is as indicated above
   by shading in the joint probability matrix: Always decide that the input was a 0. It follows that the decision
   is in error only when a 1 is transmitted. Thus, \( P_{e,\text{MAP}} = \Pi_1 < 1 - \alpha \).
   (c) The ML and MAP decision rules are the same if and only if the following two conditions hold:
   \[
   \Pi_0 \alpha > \Pi_1 (1 - \alpha) \iff \Pi_1 < (1 - \alpha) \]
   and \( \Pi_0 (1 - \alpha) > \Pi_1 \alpha \), which together imply that \( 1 - \alpha < \Pi_0 < \alpha \).

3. All three probabilities can be computed from the given information:
   \[
P(A^c \cup B^c) = 1 - P(A \cap B) = 1 - P(B|A)P(A)
   \]
   \[
P(B^c | A^c) = 1 - P(B|A^c) = 1 - P(A^c \cap B)/P(A^c) = 1 - \frac{P(B) - P(A \cap B)}{1 - P(A^c)}.
   \]

4. Let \( (1+\alpha)/2 \) and \( (1-\alpha)/2 \) denote the probabilities of winning the first and third sets. There are six possible
   outcomes as shown in the table below where the probabilities are also given.

   \[
   \begin{array}{|c|c|c|c|c|c|c|}
   \hline
   \text{Outcome} & \text{Probability} & \text{Win Match?} & \text{Value of } X & \text{Value of } Y \\
   \hline
   WW & (1+\alpha)/4 & Y & 1 & 2 \\
   WLW & (1-\alpha^2)/8 & Y & 1 & 3 \\
   LWW & (1-\alpha)^2/8 & Y & 2 & 3 \\
   LL & (1-\alpha)/4 & N & 0 & 2 \\
   LWW & (1-\alpha^2)/8 & N & 2 & 3 \\
   WLL & (1-\alpha)/4 & N & 1 & 3 \\
   \hline
   \end{array}
   \]

   (a) Hence, \( P(\text{win match}) = \frac{2 + 2\alpha + 1 - \alpha^2 + 1 - 2\alpha + \alpha^2}{8} = \frac{1}{2} \).
   (b) \( E[X] = 1 \times (1+\alpha)/4 + (1-\alpha^2)/8 + (1+\alpha)^2/8 + 2 \times (1-\alpha^2)/8 + (1-\alpha)^2/8) = 1 \).
   (c) \( E[Y] = 2 \times (1+\alpha)/4 + (1-\alpha)/4 + 3 \times [1 - \{(1+\alpha)/4 + (1-\alpha)/4\}] = 2 \times (1/2) + 3 \times (1/2) = 5/2 = 2.5. \)
   Note that the answers do not depend on the value of \( \alpha \).